Soft-Gluon Resummation for Multiparton Hard-Scattering Processes: One-Particle Inclusive Cross Section and Heavy-Quark Pair Production at Hadron Colliders

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Abstract

Quantum Chromodynamics (QCD) is nowadays accepted as the theory describing strong interactions in terms of quark and gluon degrees of freedom. A peculiar feature of the theory is asymptotic freedom, which implies that the QCD coupling constant becomes small at high energy scales. Asymptotic freedom ultimately offers the possibility to compute observables for hard-scattering processes as an expansion in the QCD coupling. When the observable is sufficiently inclusive, these computations provide reliable predictions. On the contrary, when different energy scales are involved, the convergence of the perturbative expansion is spoiled by the appearance of large logarithmic terms, which are related to the infrared behaviour of the theory. Methods exist to evaluate these logarithmically enhanced terms to higher orders in perturbation theory, and, ultimately, to sum them to all orders. The (re)summation of the large logarithmic terms extends the applicability of the QCD perturbative approach to important observables measured at high-energy colliders. Resummed calculations are thus essential for physics studies within and beyond the Standard Model at the Large Hadron Collider (LHC). This thesis deals with a class of processes which involves four partons (quark or gluons) at the lowest perturbative order. Examples of such observables are the one-particle inclusive cross section at high transverse energies and the transverse-momentum spectrum of heavy-quark pairs produced at the LHC. The thesis investigates these two processes with an explicit computation of the logarithmically enhanced terms and presents their resummation to all perturbative orders.

Contents

Chapter 1

Introduction

The Standard Model (SM) of the elementary particles is nowadays accepted as the theory of the electromagnetic, weak and strong interactions. The electroweak sector is described by the Glashow-Weinberg-Salam $SU(2) \otimes U(1)$ model [1–3], which extends Quantum Electrodynamics (QED) [4–7] to weak interactions, and the strong sector is described by Quantum Chromodynamics (QCD) [8–10]. The past fifty years have witnessed a continuous progress in the theoretical understanding of the SM and in the development of analytical and numerical tools which have paved the way to reliable quantitative predictions. A wide class of observables is nowadays understood in the framework of the SM, in the sense that their theoretical predictions agree with the corresponding measurements keeping into account theoretical and experimental uncertainties [11]. The impressive collection of such measurements and predictions has led to a precise determination of all the parameters of the model and the recent discovery [12,13] of a scalar resonance consistent with the long sought Higgs boson [14,15] indeed crowns the SM as the successful theory of elementary particle interactions.

The perturbative approach to Quantum Field Theory, i.e. the possibility to organise any calculation in terms of Feynman diagrams, has played a key role in the success of the SM. Under the assumption of small coupling constants, a first estimation of the probability amplitude for a given reaction can be obtained by taking into account only the lowest order in the perturbative expansion. This is the Leading-Order (LO) approximation, which in most cases gives the order of magnitude of the cross section for the reaction, once the squared amplitude is integrated over the available phase space of the kinematical degrees of freedom. A more precise estimate can be achieved by including the Feynman diagrams contributing to higher powers of the couplings with respect to the Born level. By following this approach any calculation can be systematically organised order-by-order in perturbation theory. Higher-order terms in the expansion should reduce the theoretical errors and yield more reliable predictions. Higher-order has thus become a synonym for higher precision and a big effort has been devoted to calculate perturbative corrections to all the relevant observables, to the extent that nowadays general methods exist to perform perturbative computations up to the Next-to-Leading Order (NLO) in the QCD coupling α_S. Analogous computations are being performed up to NLO in the EW coupling α. For some reactions even Next-to-Next-to-Leading Order (NNLO) predictions are available (see e.g. the SM review articles in [11]).

The computation of higher-order Feynman diagrams involves the inclusion of real and virtual corrections and is characterised by different kinds of singularities. Ultraviolet (UV) singularities appear only in virtual diagrams and are removed by renormalisation, i.e. by connecting all the divergent bare parameters of the Lagrangian of the SM to a set of physical observables. Infrared (IR) soft and collinear divergencies, appearing in theories with massless particles, like QED and QCD, cancel out when summing over all the degenerate physical states. Degenerate states are those with the same kinematical configuration, once soft and collinear particles are removed. In the case of QED, the Bloch-Nordsieck theorem [16] states that IR divergencies cancel out in transition probabilities for inclusive processes. Order by order in perturbative QED, the sum of the virtual and real corrections is IR finite. For a given scattering process, the IR singularities arising from virtual corrections to the elastic process and due to divergent loop integrals cancel against the IR singularities due to phase space integrals of the real corrections to the inelastic process.

QCD features a more complicated pattern of singularities. The Bloch-Nordsieck cancellation holds in the case of fully inclusive processes, with no partons in the initial state and no registered partons in the final state. The total cross section for e^+e^- annihilation belongs to this class of observables and the QCD corrections can be derived by assuming that the sum over partonic final states equals the corresponding hadronic sum. The situation gets more involved for inclusive processes with registered partons. In this case, the inclusive partonic cross-section is affected by left-over collinear singularities. The Bloch-Nordsieck theorem is extended by the Kinoshita-Lee-Nauenberg (KLN) theorem [17, 18], which states that IR divergencies cancel when summing over all the degenerate initial and final states. Here a short digression on the parton model is in order.

Contrary to the QED case, the renormalisation group equation for QCD predicts a weaker coupling at growing energies. The renormalised strong coupling $\alpha_S(\mu^2)$ vanishes in the limit $\mu^2 \to \infty$. At energies much higher than the hadronic masses, quarks and gluons propagate as free degrees of freedom and the partonic reactions are described by the renormalised QCD Lagrangian. This behaviour, known as asymptotic freedom, justifies the perturbative treatment of QCD. At scales comparable to the hadronic masses, the partons are confined in colour-singlet bound states, the hadrons. The perturbative picture offered by QCD at hard scales must be complemented by a non-perturbative description of the hadronic currents. The connection between the perturbative and the non-perturbative regimes is realised by the factorisation theorem. According to the factorisation theorem, any hadronic cross section can be written as a convolution between a partonic cross section, that describes the partonic reaction occurring at high energies, and structure functions for the registered partons, that account for the hadronic physics running from the low scales typical of bound states up to a factorisation scale μ_F. In analogy to the renormalisation program to handle UV singularities, the IR divergencies are absorbed into bare (divergent) structure functions. The scale μ_F thus acts as a regulator of the partonic cross section, at the cost of introducing scale-dependent structure functions. The factorisation theorem of QCD further states that the structure functions can be expressed in terms of *universal* distributions, the Parton Distribution Functions (PDFs) of initial-state partons in the colliding hadrons and the fragmentation functions of final-state partons to the observed hadrons. This is of crucial importance in order to extract quantitative predictions for hadron scatterings via perturbative calculations. The non-perturbative probability of extracting an initial-state parton from a colliding hadron (or that of observing a hadron from a final-state parton) can be extracted once and for all from an appropriate set of experimental measurements.

The theory also predicts the scale dependence of the parton densities. Since the hadronic cross section must be independent of the factorisation scale, the dependence of the partonic cross section and the PDFs on this scale must cancel out in their convolution. It follows that the scale evolution of the parton densities and the fragmentation functions can be inferred from perturbative QCD arguments, even if the functions themselves are non-perturbative. This approach leads to the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations, a set of coupled differential equations describing the scale evolution of the probability densities of quarks, antiquarks and gluons inside the hadrons [19–21]. Their solutions determine the parton densities at any scale in terms of the measured non-perturbative initial conditions.

The hadronic cross section is actually dependent on the factorisation scale, as well as on the renormalisation scale, once the perturbative series is truncated at a certain order in $\alpha_{\rm S}$. By varying the factorisation scale one can keep this dependence under control and assess the theoretical error. At the partonic level, the factorisation scale acts as a regulator of the collinear divergency, at the price of introducing logarithms of μ_F^2/Q^2 in the partonic cross section, where Q^2 is the energy scale of the hard process. For this reason the two scales should have the same order of magnitude. Otherwise large logarithms would enhance terms suppressed by the small coupling $\alpha_{\rm S}(Q^2) \ll 1$ and the underlying perturbative hypothesis would not be valid anymore. With an appropriate choice of the renormalisation and the factorisation scales, even processes with identified hadrons in the initial or final states are perturbatively computable, as long as one hard-momentum scale is present.

Many interesting observables unfortunately fall out of this category and the factorisation theorem is not enough to understand hard processes in terms of perturbative partonic scatterings. In the case of semi-inclusive quantities, when more than one hard scale is involved, logarithms of ratios of the hard scales are present, originating from energy or momentum thresholds. In the phase space regions where these scales are strongly ordered, the logarithmic terms are large and the validity of the perturbative approach should be questioned: in such phase space regions the theoretical predictions deviate significantly from the available experimental results.

The issue of large logarithmic terms spoiling the convergency of the perturbative series affects all the infrared sensitive quantities in theories with massless particles, like QED and QCD. This issue essentially lies in the cancellation of real and virtual infrared divergencies. Soft or collinear real radiation is undetected below the finite resolution of the experimental devices, and the theoretical definition of any observable must then be inclusive on such infrared contributions to have a physical sense. The singular structure of real and virtual corrections has to be the same and opposite in sign, in order to cancel all the divergent terms. However, the kinematics of real radiation differs from that of virtual radiation and this leads to an imbalance in the resulting infrared-safe distribution, at exclusive boundaries of the phase space. The resolution-cuts indeed introduce a logarithmic dependence in the coefficients of the perturbative expansion, in terms of extra scales that differ from the hard scale of the inclusive partonic process. This dependence is integrated out only when considering fully-inclusive quantities, but it is critical in semi-inclusive distributions. At specific phase-space boundaries where the real radiation is strongly suppressed (or strongly enhanced) the logarithmic terms are large and dominate the perturbative expansion. These are known as Sudakov effects [22]. If x is the kinematical variable that measures the distance from the exclusive boundaries, then $x \to 0$ denotes the Sudakov region. In the perturbative series there are tipically up to two powers of $L = \ln(x)$ for each power of α_S, one of soft and one of collinear origin. The leading term in the near-threshold region at the first perturbative order is then proportional to $\alpha_S L^2$, the next-to-leading term is proportional to $\alpha_S L$. At all orders in α_S, the perturbative coefficients involve terms of the type $\alpha_S^n L^m (m \leq 2n)$, which are not suppressed as ordered powers of α_S.

The solution to the problem was originally proposed in QED [23] for the energy distribution of infrared radiation in e^+e^- collisions. When the radiated energy ω is small in comparison to the transferred energy E, the behaviour of the distribution is governed by the exponential $(\omega/E)^\alpha$. The large logarithmic terms are shown to be universal and proportional to the lowest order (Born-level) cross section and they are 'resummed' in the exponent, to all-order in the QED coupling α.

The same approach has been successfully extended to QCD, leading to an important progress in the phenomenology of high-energy hadron-hadron collisions. A typical process sensitive to soft-gluon effects is the hadroproduction of a colourless system of large mass Q (a W or Z boson, an excited γ, a muon pair) at measured transverse momentum q_T. The gluons radiated from the annihilating partons carry away transverse momentum and at low Q the observed q_T-distribution reflects the transverse momentum distribution of the incoming partons. As Q is made larger, the gluon radiation increases and the q_T-distribution broadens. With the advent of QCD factorisation theorem, it was possible to compute the transverse momentum distribution in the Drell-Yan (DY) process up to the NLO corrections in α_S [24]. The result, obtained through a plain perturbative approach, is only useful at $q_T \sim Q$, because of the presence of logarithms $L_T = \ln(Q^2/q_T^2)$ in the perturbative coefficients. As $q_T \to 0$ the fixed order cross section is indeed divergent. The low transverse-momentum region $q_T \ll Q$ is a Sudakov region for this observable, where soft radiation is suppressed and the logarithms become large. In [25–27] it was shown that the leading terms $\alpha_S^n L_T^{2n-1}/q_T^2$ came from dressed q_T-ordered ladder diagrams in a physical gauge (see Section 2.1), where each subsequent emission is soft or collinear (hence q_{T} is low). Thus, they could be calculated and exponentiated to all-order in α_{S}. The formula proposed by Collins, Soper and Sterman (CSS) in [28], with PDFs evaluated at the soft scale $1/b$ (see Chapter 5) paved the way to the systematic analysis of Sudakov effects at low measured transverse momentum in QCD. The modern transverse-momentum resummation formalism [29] is an extension of the CSS formula to the rich phenomenology of final states produced at hadron colliders.

Another class of soft-gluon sensitive observables is represented by inclusive hard-scattering cross sections in kinematical configurations that are close to the partonic threshold. Typical examples are the Deep Inelastic Scattering (DIS) cross section and DY-like cross sections for the production of lepton pairs and Higgs or massive vector bosons. The first attempts to calculate large logarithmic terms to all orders in semi-inclusive limits can be found in [30–32]. The complete resummed formulae with next-to-leading logarithmic accuracy for the Drell-Yan and DIS cross sections first appeared in [33], followed shortly after by the independent result of [34–36]. The same techniques were then applied to develop the coherent branching algorithms for NLL Monte Carlo [37] and the resummation of distributions characterising the structure of the hadronic final states (see e.g. Ref. [38] and references therein).

The many calculations available in the literature demonstrate how important the concept of resummation is to the phenomenology of Hadron Physics. Resummation-improved predictions successfully account for the experimental results in the critical boundaries, where simple fixed-order results fail badly. When only two QCD partons take part to the Born-level scattering, the contribution due to large-angle soft-gluon exchange between the annihilating partons cancels out at the level of squared amplitudes: the large logarithmic terms to be resummed correspond to phase-space regions of (soft and hard) collinear emission and are free of colour correlations. The resulting colour flow is equivalent to that of independent self-interactions of the hard partons via the emission and reabsorption of real and virtual soft gluons. This is the case of the hadroproduction of Higgs bosons, massive vector bosons and in general of colourless composite systems produced in DY-like processes. When QCD partons are produced at the Born level, instead of a colourless system, the same logarithmic structure typical of DY-like emission is to be found in the amplitude, along with additional terms due to soft-collinear QCD radiation from the final-state hard partons. The squared amplitude now receives contributions from soft-gluon exchanges between all the hard partons, and the colour flow is more involved: due to the non-abelian properties of the QCD interactions, the matrix elements do not respect a simple scalar factorisation anymore, and the kinematical factorisation can be preserved only by introducing colour operators and colour amplitudes on an abstract colour space (see Section 2.5). The collinear functions that resum DY-like logarithms are diagonal in colour space, while the additional logarithmic terms are proportional to colour operators projected on the colour states of the LO kinematics. The extension of the known resummation formulae to include these non-collinear effects and the ensuing colour correlations is not trivial and requires new exponential functions defined in colour space. The general soft-gluon resummation formalism for inclusive cross sections in these complex multiparton processes was developed in a series of papers [39–44]. In recent years, techniques and methods of Soft Collinear Effective Theory [45–47] (SCET) have also been developed and applied to resummation for inclusive cross sections near (partonic) threshold [48–54] and to transverse-momentum resummation [55–63].

This thesis is devoted to the study of two different processes initiated by the hard scattering of four QCD partons: the one-particle inclusive production at large transverse momentum in hadronic collisions and the production of a heavy quark pair at small transverse momentum. By using the universality of soft and collinear emissions, we compute the structure of the logarithmically enhanced contribution at relative order $\mathcal{O}(\alpha_S)$ for these two processes, up to the subleading (constant) terms. Our results are factorised in colour space and this allows us to explicitly disentangle colour-interference effects. Inspired by the BCMN approach [44], we then derive all-order resummation formulae, valid at arbitrary logarithmic accuracy and written in terms of collinear radiative factors and of colour-space radiative factors. The former are already known: being associated to individual radiators, they are process-independent and they can be extracted from the resummed results for the hadroproduction of colourless final states with high invariant mass and small transverse momentum. The latter exponentiate 'soft anomalous dimensions' that take into account soft-gluon radiation at large angles, with the ensuing colour correlations. Thanks to our process-independent NLO results we explicitly determine all the resummation coefficients up to NLL accuracy, including the soft anomalous dimensions in colour space. Furthermore, from the colour structure of the constant terms at relative order $\mathcal{O}(\alpha_S)$, we are able to extract the explicit form of (IR finite) one-loop hard-virtual amplitudes at the same perturbative order. From this ingredient we can determine an entire class of resummed contribution at NNLL accuracy, as discussed in Chapters 4 and 6.

The rest of the thesis is organised as follows. In Chapter 2 we discuss the separation of the perturbative and non-perturbative regimes in QCD and the factorisation of mass singularities. We discuss the DGLAP evolution equations and show how the large logarithmic terms arise in soft and collinear limits. The all-order exponentiation of the leading logarithmic terms is treated in general terms, by using the factorisation properties of soft and collinear radiation. We also introduce the abstract colour space where the factorisation of the singular terms can be formulated in a process-independent way. Explicit resummation formulae are presented in Chapter 3. The analysis of the Drell-Yan process and of the deep inelastic scattering in the partonic-threshold limit leads to the definition of process-independent semi-inclusive form-factors. These factors capture the effect of collinear radiation and resums the large logarithmic terms to all orders in perturbation theory. The same formalism can be applied to any process in the kinematical region of partonic threshold, when up to three QCD partons are involved, since the effect of non-collinear radiation is suppressed and the colour algebra obeys a scalar factorisation. In Chapter 4 we consider the cross section for the one-particle inclusive production and we calculate the NLO corrections in the threshold limit in which the final-state system that recoils against the triggered parton is constrained to have a small invariant mass. We then present our all-order resummation formula, which explicitly resums all NLL terms and is organised in a way that makes the extension to the NNLL accuracy simple. The tranverse-momentum resummation formalism of Ref. [29] is presented in Chapter 5 and applied to the cross section for the production of a colourless system of high total invariant mass, in the kinematical region where the transverse-momentum of the final-state system vanishes. The large logarithms of collinear origin are resummed by universal Sudakov form-factors, while any contribution due to large-angle emission is suppressed in the low-q_{T} region. The colliding initial-state partons are the only QCD particles involved in the scattering and the colour flow is described by scalars. In Chapter 6 we present the resummed transverse-momentum distribution for the production of a heavy-quark pair. The cross section is calculated in the low-q_{T} limit, up to NLO, and the fixed order result is compared with that of the previous Chapter. By using the same techniques as in Chapter 4, we propose our extension of the resummation formalism to include colour-correlation terms due to large-angle soft-gluon emissions. We explicitly calculate all the resummation coefficients at NLL accuracy and we discuss the extension to NNLL accuracy, including the issue of spin and azimuthal correlations for the gluon fusion channel [64].

Chapter 2

Factorisation and soft gluon effects

The description of QCD radiation in soft and collinear regimes and the all-order calculation of the ensuing large logarithmic contributions is made possible by the universal factorisation properties of infrared radiation. Here we review the techniques that lead to the process-independent description of collinear splittings and soft-gluon emission. The resummed formulae are essentially build upon the same approximations.

The description of collinear radiation can be derived from the perturbative behaviour of the parton distribution functions, as dictated by the DGLAP evolution equations. Soft-gluon effects can be better understood as a generalisation of simpler (abelian) soft-photon emission. The derivation of the DGLAP equations and a review of QED resummation is hence in order.

2.1 DGLAP equations

The parton model is first introduced within the simple kinematics of (unpolarised) lepton-hadron deep inelastic scattering (DIS) and bare parton densities $f(x)$ are defined in the Bjorken limit of infinite momentum transfer. The Born-level process is free of strong interactions. It is therefore the ideal framework to focus on the hadron undergoing scattering and to introduce higher-order QCD effects.

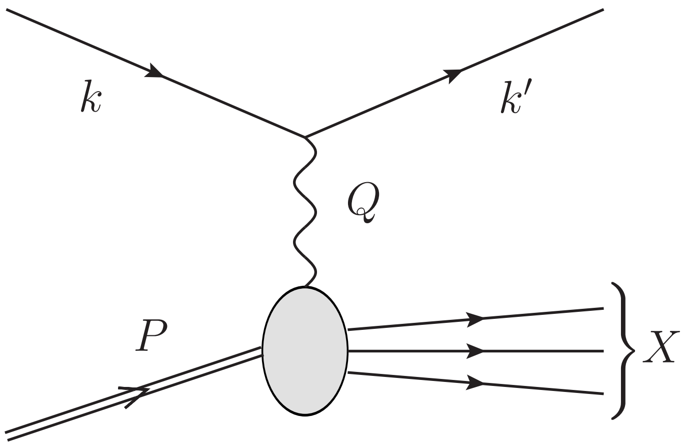

Figure 2.1: Deep Inelastic Scattering.

The hadronic reaction is a scattering process between a lepton l, with initial-state momentum k and final-state momentum k', and a hadron h of momentum P, undergoing fragmentation into a hadronic final-state X,

$$l(k) + h(P) \to l(k') + X\,. \tag{2.1}$$

The squared centre-of-mass energy is $S = (k + P)^2$ and the (space-like) transferred momentum is $Q = k' - k$. We define Q^2 such that $Q^2 = -Q_\mu Q^\mu > 0$. The inelastic regime is characterised by an invariant mass squared of the hadronic final state, $W^2 = (P + q)^2$, much bigger than the mass squared $P^2 = M_h^2$ of the incoming hadron. It follows that

$$W^2 = M_h^2 + 2P \cdot Q - Q^2 > M_h^2 \quad \Rightarrow \quad x = \frac{Q^2}{2P \cdot Q} < 1\,. \tag{2.2}$$

The dimensionless variable x is the Bjorken variable, of real values in the range $(0, 1)$. The limit $x \to 1$ corresponds to the elastic regime $W^2 = M_h^2$. Due to Lorentz invariance and gauge invariance the leading-order cross section for the exchange of a virtual photon at fixed Q^2, x can be written as

$$\frac{d\sigma}{dQ^2 dx} = \frac{2\pi\alpha^2}{xQ^2}\Big\{\left[1 + (1-y)^2\right] F_2(x, Q^2) - y^2 F_L(x, Q^2)\Big\}\,, \tag{2.3}$$

where α is the electroweak coupling and y is the energy fraction transferred by the scattered lepton, as measured in rest frame of the hadron,

$$y = \frac{Q^2}{xS} = \frac{2P \cdot Q}{S}\,. \tag{2.4}$$

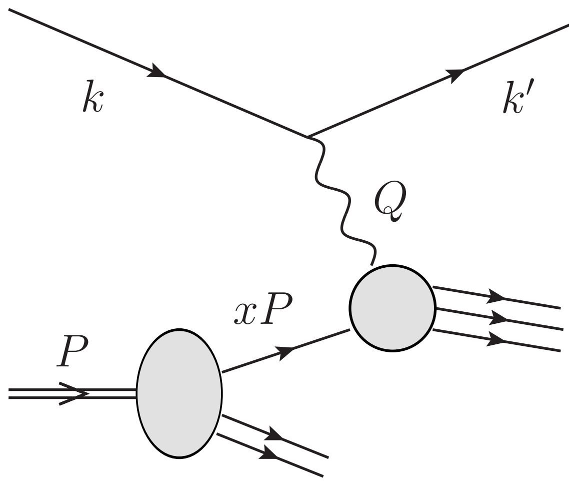

Figure 2.2: The Deep Inelastic scattering as a partonic reaction.

The dimensionless *proton structure functions* F_2, F_L of the incoming hadron parametrise the non-perturbative information associated to the strong scattering dynamics.

We are interested in a *deep* scattering, with momentum transfer Q^2 much higher that the hadron mass M_h^2 and in the limit where the Bjorken variable x is constant. In such a limit the strong coupling $\alpha_S(Q^2)$ is small enough to proceed with the perturbative approach. According to asymptotic freedom, the partons inside the hadron are essentially free. In addition, if the parton density in the hadron is not too large, the virtual photon which mediates the scattering will be able to interact with only a single constituent of the hadron. In the DIS regime this assumption is reasonable. The higher is the momentum transfer Q, the shorter is the time in which the interaction takes place. In the parton model, the inelastic electron-hadron scattering is an *elastic* electron-quark scattering, at the Born level (Figure 2.2). The interactions of the partons among themselves, occurring before or after the hard scattering, cannot interfere with it. The two processes are thus incoherent and the total cross section for DIS is computed by combining probabilities, rather than amplitudes. The probability associated to the electron-quark scattering is its (perturbative) cross-section. The probability that the photon encounters a free parton of flavour a and momentum $p = xP$ is the (bare) parton density $f_{a/H}(x)$. The structure functions are equal to the convolution between parton densities and *parton* coefficient functions,

$$F_i(x, Q^2) \sim \sum_{a=q,\bar{q}} \int_x^1 dz\, f_{a/h}(z)\, \mathcal{F}_{i,a}(x/z, Q^2)\,. \tag{2.5}$$

The argument x/z of the parton coefficient function in the right-hand-side of Eq. (2.5) reflects the momentum fraction of the parton with respect to the parent hadron, fixed by the argument z of the parton density. The coefficient functions $\mathcal{F}_{i,a}$ are defined from the partonic cross section, by the same relation (2.3) that connects the proton structure function F_i and the hadronic cross section. The hadron structure is entirely parametrised by the parton densities and the residual information is that of an *elastic* scattering between elementary (point-like) QCD states. The explicit computation of the matrix element for $\gamma^* q \to \gamma^* q$ scattering with an on-shell massless quark leads to

$$F_2(x, Q^2) \sim F_2(x) = x \sum_{a=q,\bar{q}} e_a^2 \, f_{a/h}(x) \,, \qquad F_L(x, Q^2) \sim 0 \,. \tag{2.6}$$

According to the parton model, the structure functions are independent of Q^2 in the large Q^2 limit, since the parton densities are independent of the *hard* scale Q^2. This behaviour, known as Bjorken scaling, is due to the point-like nature of the hadron constituents and it is confirmed by the experimental results. Moreover the parton model predicts the Callan-Gross relation, i.e. a vanishing longitudinal structure function F_L in the large Q^2 limit for on-shell, massless, spin 1/2 partons. The interaction of such particles with the longitudinal degrees of freedom of virtual photons would violate the helicity conservation. These results confirm the validity of the (leading-order) parton model and of measurable parton densities. Higher order QCD corrections violate both the Bjorken scaling and the Callan-Gross relation. The factorisation picture is still valid, as long as the parton densities are promoted to scale-dependent functions.

A better approximation of the structure functions can be derived from Eq. (2.5) by employing NLO matrix elements for the partonic scattering. The $\mathcal{O}(\alpha_S)$ corrections to the squared amplitude include the sum squared of one-loop virtual contributions as well as the sum squared of the real emission diagrams, where one final-state gluon is radiated from the initial- and the final-state quark. The sum of the virtual and the real contributions (Figure 2.3), the latter integrated over the gluon phase-space, leads to the cancellation of soft and final-state collinear singularities. In agreement to the KLN theorem, an initial-state collinear pole survives, due to the *exclusive* nature of the partonic cross section with respect to the specific initial-state parton taking part to the scattering. At the hadronic level, the parton densities imply a sum over all the possible initial-states and the residual collinear singularities must cancel. In order to cure the UV divergencies, the parameter of the Lagrangian are considered divergent. The renormalisation program

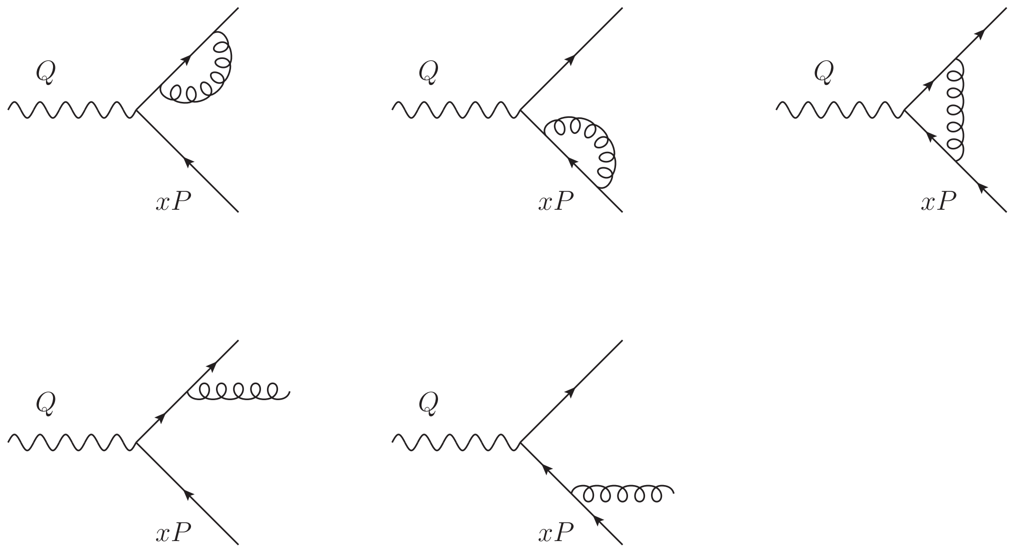

Figure 2.3: Feynman diagrams of the virtual (first row) and real (second row) NLO QCD corrections to the $\gamma^* q$ scattering.

consists in consistently reabsorbing the UV poles into the 'bare' parameters and to connect the 'dressed' parameters to physical observables, order-by-order in perturbation theory. In a similar fashion, the parton densities of the naive parton model can be dressed with the long-range effects due to collinear splittings from the hadronic scale $M_h^2 \sim 0$ up to a perturbative scale μ_F^2. The higher-order collinear poles are then reabsorbed into the higher-order corrections to the parton densities. The factorisation scale μ_F separates long- and short-range (soft- and hard-scale) effects, acting as a regulator of the partonic cross section.

In the region of interest, the emission of a gluon of momentum q from the initial-state parton can be described via a collinear approximation, where the transverse-momentum squared q_{T}^2 of the gluon (transverse with the respect to the longitudinal direction p) is small with respect to the hard scale Q^2. Under this approximation and in a physical gauge in which the gluon admits only transverse polarisations, the interference term of initial- and final-state radiation is convergent. The only collinear-divergent contribution is due to the square of the initial-state radiation matrix element. The dominant term (in the region $q_{\mathrm{T}}^2 \ll Q^2$) is proportional to the Born-level amplitude squared $|\mathcal{M}_{\mathrm{DIS}}|^2$,

$$\frac{\alpha_{\mathrm{S}}}{2\pi}\frac{dz}{z}P_q^{(1)}(z)\,\frac{dq_{\mathrm{T}}^2}{q_{\mathrm{T}}^2}\big(1+\mathcal{O}(|q_{\mathrm{T}}|)\big)\;|\mathcal{M}_{\mathrm{DIS}}(p,q)|^2\,. \tag{2.7}$$

The probability of emission of a single gluon from a quark (antiquark) is described by the LO regularised diagonal Altarelli-Parisi (AP) splitting probability $P_q^{(1)}(z)$, where z is the energy fraction left to the quark after splitting. 'Regularised' refers to the inclusion of the divergent contact term at $z = 1$, from the self-energy diagram of the quark. This virtual contribution is required in order to define a regular distribution over the available energy phase-space. 'Diagonal' refers to flavour conservation during the splitting. The same factorisation property is valid for a triple-gluon vertex and for non-diagonal splittings as well. The same arguments can be straightforwardly extended to any initial-state parton evolving to the required flavour after splitting (a quark or antiquark in the case of DIS off a vector boson). The AP probability describing the evolution of a 'parent' parton b into a 'child' parton a carrying a longitudinal momentum fraction z is denoted by $P_{ab}(z, \alpha_S)$. These functions are universal, they only depend on the QCD interactions between gluons, quarks and antiquarks. They admit a perturbative expansion in α_S,

$$P_{ab}(z,\alpha_S) = \frac{\alpha_S}{2\pi} P_{ab}^{(1)}(z) + \sum_{n=2}^{\infty} \left(\frac{\alpha_S}{2\pi}\right)^n P_{ab}^{(n)}(z) \,. \tag{2.8}$$

The coefficient $P_{ab}^{(1)}$ describes the energy spectrum of the splitting due to the emission of a single parton and the coefficients $P_{ab}^{(n)}$ include higher-order effects due to multiple QCD radiation. The explicit calculation of the LO coefficients [21] of the regularised probabilities gives

$$P_{qq}^{(1)}(z) = C_F \left[\frac{1+z^2}{1-z}\right]_+ , \tag{2.9}$$

$$P_{qg}^{(1)}(z) = T_R \left[z^2 + (1-z)^2\right] , \tag{2.10}$$

$$P_{gq}^{(1)}(z) = C_F \left[\frac{1+(1-z)^2}{z}\right] , \tag{2.11}$$

$$P_{gg}^{(1)}(z) = 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \delta(1-z)\beta_0 \,, \tag{2.12}$$

where $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$, $T_R = 1/2$ are the $SU(N_c)$ colour factors. The corresponding Feynman diagrams are listed in Figure 2.4 and the results are identical in case of an antiquark. The coefficient $\beta_0 = 11/6\, C_A - 1/3\, n_f$, where n_f is the number of massless quarks, is the first coefficient of the QCD β-function. The 'plus-distribution' of a function $f(z)$ is defined at the integral level as

$$\int_0^1 dz\, [f(z)]_+ \, g(z) = \int_0^1 dz\, [f(z) - f(1)]\, g(z) \,, \tag{2.13}$$

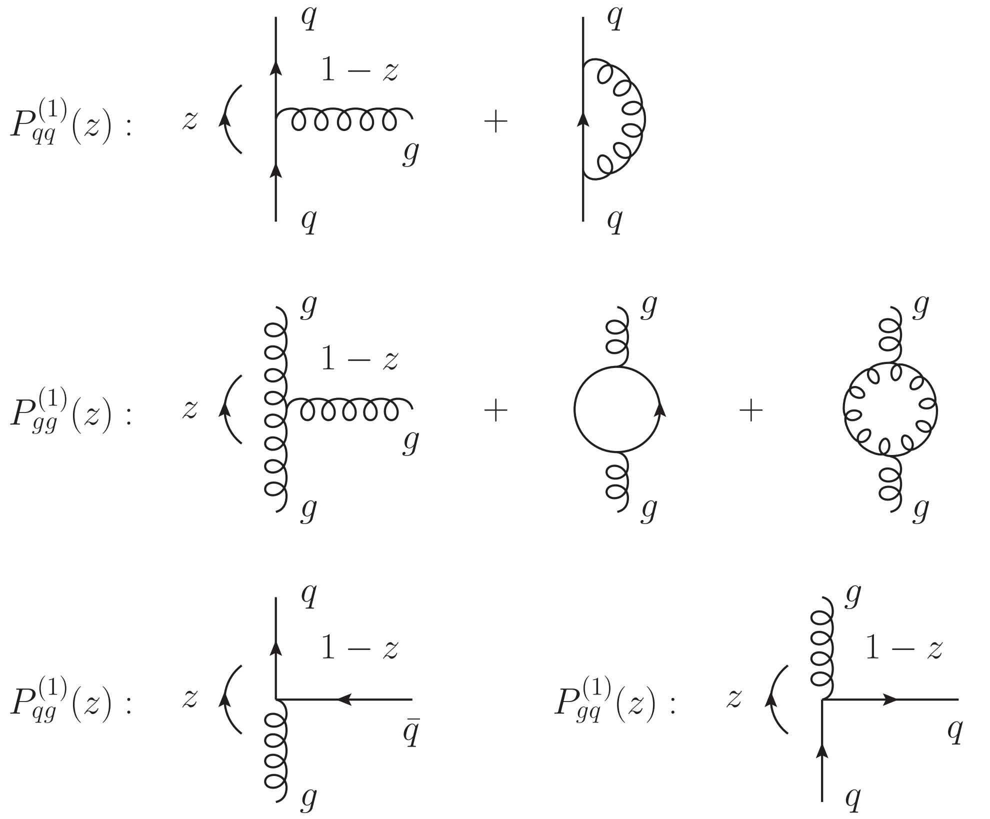

Figure 2.4: Born-level diagrams contributing to the regularised splitting functions.

where $g(z)$ is a test function, regular at $z = 1$. In Eq. (2.9) and in Eq. (2.12) the $f(z)$ functions of Eq. (2.13) are proportional to $1/(1-z)$, logarithmically divergent at $z = 1$, while in the regularised distribution the pole at $z = 1$ is cancelled via the subtraction of the contact term $f(1)$. This term accounts for the virtual emission, proportional to a $\delta(1-z)$ distribution. The real part of the AP probabilities, corresponding to the real diagrams of Fig. 2.4, would directly contain the divergent functions and not their regularised versions. The divergent energy spectrum $dz/(1-z)$ is the typical bremsstrahlung spectrum for the emission of a massless gauge boson (gluon in QCD or photon in QED). In this case, the pole at $z = 1$ is associated to the emission of a soft gluon with energy $1 - z \to 0$. The difference $[f(z) - f(1)]$ that cancels the divergence is due to a compensation between the real and the virtual emission. The $\delta(1 - z)$ and the plus-distributions in the quark-quark and in the gluon-gluon splitting probabilities are due to the inclusion of the quark and gluon self-energy diagrams, respectively.

The integral of (2.7) over the available transverse-momentum phase-space features a pole at $q_\mathrm{T}^2 = 0$, which can be regularised e.g. by employing dimensional regularisation with a number of dimensions $d = 4 - 2\epsilon \to 4^+$. Explicitly, up to a scale μ^2,

$$\mu_R^{2\epsilon} \int_0^{\mu^2} dq_\mathrm{T}^2 \, (q_\mathrm{T}^2)^{-1-2\epsilon} = -\frac{1}{\epsilon} \left(\frac{\mu_R^2}{\mu^2} \right)^\epsilon , \tag{2.14}$$

where μ_R^2 is the renormalisation scale. This is the collinear pole to be reabsorbed into the bare parton density. The parton density regularised at a scale μ_F^2 is defined, at the first order in α_S and up to finite terms, as

$$f_{a/h}(x, \mu_F^2) = f_{a/h}(x) - \frac{\alpha_\mathrm{S}(\mu_F^2)}{2\pi} \frac{1}{\epsilon} \left(\frac{\mu_R^2}{\mu_F^2} \right)^\epsilon \sum_b \int_x^1 \frac{dz}{z} P_{ab}^{(1)}(z) \, f_{b/h}(x/z) + \mathcal{O}(\alpha_\mathrm{S}^2). \tag{2.15}$$

The specification of the finite terms varies with the factorisation scheme employed. Together with the $1/\epsilon$ pole, the parton density in Eq. (2.15) includes any single-gluon emission with transverse-momentum spectrum (2.14), within the scale $\mu^2 = \mu_F^2$. This is the quantity of interest, whereas $f_{a/h}(x)$ does not have any physical meaning. The observable to be directly measured at the colliders is the hadron structure function F_2. Its expression beyond the LO in α_S is

$$F_2(x, Q^2) = \sum_{a=q,\bar{q}} \sum_{b=q,\bar{q},g} \int_x^1 dz \, f_{b/h}(z, \mu_F^2) \, \mathcal{F}_{2,ab} \left(\alpha_\mathrm{S}(\mu_R^2); x/z, Q^2; \mu_F^2 \right) . \tag{2.16}$$

Eq. (2.16) extends the validity of Eq. (2.5) to higher-order corrections, by making use of the improved parton densities. The parton b, which takes part to the perturbative process with an initial momentum zP, can now be a quark, an antiquark or a gluon. The underlying assumption is that Eq. (2.16), proven at the first non-trivial order, can be extended to all orders. If this holds, all the ϵ poles of collinear origin can be reabsorbed into the bare parton density, along the lines of Eq. (2.15).

The meaning of Eq. (2.16) is that perturbative QCD itself can not predict a hadronic observable like F_2, but only its normalisation with respect to the structure functions measured in other scattering processes. The perturbative information lies in $\mathcal{F}_{2,ab}$. The infrared behaviour at the NLO is described by the single splitting (2.7), with a general (non-diagonal) AP probability P_{ab}. The transverse-momentum spectrum of (2.14) from μ_F^2 to $\mu^2 \sim Q^2$ is part of the perturbative partonic cross section and gives the logarithmic term

$$\int_{\mu_F^2}^{Q^2} \frac{dq_\mathrm{T}^2}{q_\mathrm{T}^2} = \ln \frac{Q^2}{\mu_F^2} , \tag{2.17}$$

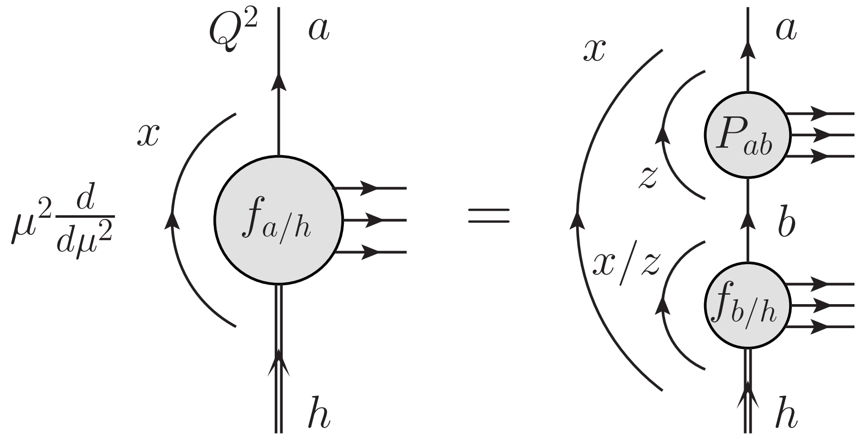

Figure 2.5: Pictorial representation of the DGLAP evolution equations.

which enters the expression of the coefficient function $\mathcal{F}_{2,ab}$:

$$\mathcal{F}_{2,ab}(\alpha_S; z, Q^2; \mu_F^2) = z\, e_a^2 \left[\delta_{ab}\delta(1-z) + P_{ab}(\alpha_S; z) \ln \frac{Q^2}{\mu_F^2} + R_{ab}(\alpha_S; z, Q^2) \right], \tag{2.18}$$

$$R_{ab}\left(\alpha_S; z, Q^2\right) = \frac{\alpha_S}{2\pi} R_{ab}^{(1)}(z, Q^2) + \sum_{n=2}^{\infty} \left(\frac{\alpha_S}{2\pi}\right)^n R_{ab}^{(n)}(z, Q^2)\,. \tag{2.19}$$

The lowest-order term in the r.h.s. of Eq. (2.18) is consistent with Eq. (2.6): the parton b that enters the perturbative QED reaction has the same flavour a and momentum fraction z as provided by the parton density. At higher-orders in α_S, the parton b can also be a gluon and in general it carries a momentum xP, with $x < z$. The QCD corrections are described by the AP probability of Eq. (2.8) and the perturbative function of Eq. (2.19), both starting at order $\mathcal{O}(\alpha_S)$. The function R_{ab} is the sum of the non-logarithmic contributions from the real emission diagrams, due to the $\mathcal{O}(|q_T|)$ term in Eq. (2.7), and the multi-loop virtual corrections not included in P_{ab}, proportional to $\delta(1-z)$.

The parton coefficient function $\mathcal{F}_{2,ab}$ and hence the structure function F_2 now depend on the physical scale Q^2, with the consequent violation of the Bjorken scaling, in agreement with the experimental results. On the contrary, the scale μ_F^2 appears only in the right-hand-side of Eq. (2.16). The cross section for the hadronic scattering is independent of such arbitrary unphysical scale and so must be the structure function. It follows that any variation of the parton density $f_{b/h}$ due to a variation of the scale must be compensated by the opposite behaviour of the function $\mathcal{F}_{2,ab}$. This behaviour is described by Eq. (2.18). From scale invariance of F_2, $dF_2/d\mu_F^2 = 0$, it is thus possible to control the scale evolution of the parton densities, in close similarity to the Renormalisation Group Equation (RGE) approach to the μ_R^2 scaling. By differentiating the r.h.s. of Eq. (2.16) with respect to $\ln(\mu_F^2)$ and by taking the first order in α_S, one gets the DGLAP evolution equations,

$$\mu^2 \frac{d}{d\mu^2} f_{a/h}(x,\mu^2) = \sum_{b=q,\bar{q},g} \int_x^1 \frac{dz}{z} P_{ab}(x/z, \alpha_S(\mu^2))\, f_{b/h}(z,\mu^2)\,. \tag{2.20}$$

Even if the parton densities are associated to a non-perturbative process, an infinitesimal scale variation is described by a perturbative differential equation involving the AP splitting probabilities. Eq. (2.20) expresses an iterative picture, where the probability of finding a parton a of longitudinal momentum fraction x and transverse momentum μ^2 is equal to the probability of finding a parton b of longitudinal momentum z and the same transverse momentum, convoluted with the probability for the QCD splitting process from b to a, where the fraction x/z of longitudinal momentum is conserved.

The solution of the AP equations for a finite scale variation from $\mu^2 = Q_0^2$ to $\mu^2 = Q^2$ leads to the exponentiation of logarithms of the ratio Q^2/Q_0^2. The explicit result can be easily derived for the non-singlet (NS) flavour configuration. Due to a potential flavour change occurring in the splittings, Eq. (2.20) defines a set of coupled differential equations, each of them featuring a convolution integral. The equation for the NS distribution,

$$f_h^{\rm NS} = \sum_{i=1}^{n_f} (f_{q_i/h} - f_{\bar{q}_i/h})\,, \tag{2.21}$$

is instead decoupled and hence admits a simpler solution. In addition, a convolution of real functions is a simple product in Mellin space. The Mellin moment f_N of a function $f(z)$ is defined as

$$f_N = \int_0^1 dz\, z^{N-1}\, f(z)\,, \tag{2.22}$$

and the convolution integral of two functions is transformed to

$$\begin{aligned} &\int_0^1 dz\, z^{N-1} \int_0^1 dt\, f(t)\, g(z/t)\, \Theta(z-t) = \\ &\int_0^1 dt\, t^{N-1} f(t) \int_0^1 d(z/t)\, (z/t)^{N-1} g(z/t) = f_N\, g_N\,. \end{aligned} \tag{2.23}$$

The AP evolution equation for the NS-distribution f_N^{NS} in Mellin space reads

$$\mu^2 \frac{d}{d\mu^2} f_{h,N}^{\mathrm{NS}}(\mu^2) = \gamma_N^{\mathrm{NS}}(\alpha_{\mathrm{S}}(\mu^2))\, f_{h,N}^{\mathrm{NS}}(\mu^2)\,, \tag{2.24}$$

where γ_N^{NS} is the non-singlet Altarelli-Parisi probability density function in Mellin space. The Mellin transform of a splitting function is usually denoted as

$$\gamma_{ab,N}(\alpha_{\mathrm{S}}) = \int_0^1 dz\, z^{N-1}\, P_{ab}(\alpha_{\mathrm{S}}; z)\,, \tag{2.25}$$

and admits a perturbative expansion, whose coefficients are the Mellin transforms of the $P_{ab}^{(n)}(z)$ coefficients. At the lowest order the NS distribution receives contributions from the quark-quark splitting only,

$$\gamma_N^{\mathrm{NS}}(\alpha_{\mathrm{S}}) = \frac{\alpha_{\mathrm{S}}}{2\pi} \gamma_{qq,N}^{(1)} + \mathcal{O}(\alpha_{\mathrm{S}}^2)\,. \tag{2.26}$$

The integration of Eq. (2.24) is straightforward and yields to

$$f_{h,N}^{\mathrm{NS}}(Q^2) = f_{h,N}^{\mathrm{NS}}(Q_0^2)\, \exp\left\{ \int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2}\, \gamma_N^{\mathrm{NS}}(\alpha_{\mathrm{S}}(q^2)) \right\}. \tag{2.27}$$

The initial condition $f_N^{\mathrm{NS}}(Q_0^2)$ contains the information on the partonic interactions occurring from a non-perturbative scale M_h^2 to the scale Q_0^2. The exponential in the r.h.s. of Eq. (2.27) describes the subsequent evolution up to the scale Q^2. By ignoring the coupling evolution, i.e. $\alpha_{\mathrm{S}}(q^2) \sim \alpha_{\mathrm{S}}(Q^2)$, and by making use of the expansion (2.26), the evolution of the non-singlet parton density is approximately described by

$$f_{h,N}^{\mathrm{NS}}(Q^2) \sim f_{h,N}^{\mathrm{NS}}(Q_0^2)\, \exp\left\{ \frac{\alpha_{\mathrm{S}}(Q^2)}{2\pi} \ln \frac{Q^2}{Q_0^2}\, \gamma_{qq,N}^{(1)} \right\}. \tag{2.28}$$

Moreover $\gamma_{qq,N}^{(0)} \sim -2C_F \ln(N)$ in the large-N limit, corresponding to the soft-emission region $z \to 1$ of the longitudinal-momentum space, so that

$$f_{h,N}^{\mathrm{NS}}(Q^2) \stackrel{[}{N}\!\to \infty]\!\sim f_{h,N}^{\mathrm{NS}}(Q_0^2)\, \exp\left\{ -C_F \frac{\alpha_{\mathrm{S}}(Q^2)}{\pi} \ln \frac{Q^2}{Q_0^2}\, \ln N \right\}. \tag{2.29}$$

The logarithm $\ln(Q^2/Q_0^2)$ in the r.h.s. of Eq. (2.29) is potentially large and the argument of the exponential could be of order 1, even if $\alpha_{\mathrm{S}}(Q^2) \ll 1$ at the hard scale Q^2. The exponential itself can not be expanded and truncated to fixed-order without compromising the convergence of the perturbative series. Eq. (2.29) thus correctly resums the leading logarithmic terms to all orders in α_S and the leading logarithmic behaviour is controlled by the lowest-order term in the *argument* of the exponential. Eq. (2.27) formally resums all the logarithmic terms which are potentially large. Subleading $\ln(Q^2/Q_0^2)$ terms can be recovered by implementing the scale variation of the running coupling $\alpha_S(q^2)$. Subleading $\ln(N)$ terms, if needed, are connected to higher-order terms in the expansion of the γ_N^{NS}.

The physical content of Eq. (2.27) can be better appreciated in the direct space of the longitudinal momentum fraction. The AP evolution equation (2.20), valid for an infinitesimal scaling $d\mu^2$, follows from the simple splitting probability (2.7), as derived in the collinear hypothesis of small transverse momentum of the radiated gluon, $q_T^2 \ll Q^2$. As a first approximation, the evolution from a very soft scale Q_0^2 to the hard scale Q can be described by a single-splitting event, resulting in the (large) logarithmic contribution

$$\alpha_S \ln \frac{Q^2}{Q_0^2} \,. \tag{2.30}$$

At higher order in α_S, the same approximation can be effectively employed to parametrise multiple splitting, in the hypothesis of strongly q_T^2-ordered emissions. The idea is simply to generate 'ladder diagrams' of iterated collinear splittings, where the phase space available to the parton for emission $i+1$ is further constrained with respect to the previous emission i. This is the same iterative picture implied by Eq. (2.20). These diagrams do not depend on the details of the specific hard process and they have the same behaviour in all the processes, which is the reason why the parton distribution functions are universal. The leading q_T contribution of order α_S^n, in a physical gauge and in the region $Q_0^2 \ll k_{1\perp}^2 \ll k_{2\perp}^2 \ll \ldots \ll k_{n\perp}^2 \ll Q^2$, is proportional to

$$\alpha_S^n \int_{Q_0^2}^{Q^2} \frac{dk_{1\perp}^2}{k_{1\perp}^2} \int_{k_{1\perp}^2}^{Q^2} \frac{dk_{2\perp}^2}{k_{2\perp}^2} \cdots \int_{Q_0^2}^{k_{n-1\perp}^2} \frac{dk_{n\perp}^2}{k_{n\perp}^2} \sim \frac{1}{n!} \left(\alpha_S \ln \frac{Q^2}{Q_0^2} \right)^n \,. \tag{2.31}$$

The sum over n, i.e. over all the multiple-emission diagrams, naturally leads to the exponential function of argument (2.30). At subleading logarithmic orders the interference corrections to the ladder diagrams must be taken into account, but not the interference terms with the specific subprocess.

2.2 Dynamical and kinematical factorisation

We now want to discuss the conditions under which the logarithmically enhanced terms affecting fixed-order perturbative calculations near the boundary of the phase space admit an all-order resummation. In order to resum the large logarithmic terms to all orders in α_S we need to factorise and exponentiate them (cfr. Eqs. (2.61), (2.83) and see Chapter 3). The feasibility of exponentiation is subject to the dynamical and kinematical factorisation properties of the perturbative cross sections, but not every infrared and collinear safe observable can be factorised. Here we restrict ourselves to soft-gluon sensitive observables that admit a fully factorisable cross section in the Sudakov limit. As regards dynamical factorisation, it follows from the universal factorisation properties of QCD matrix elements in the soft and collinear regions [65]. In the limit in which a gluon becomes soft (see Section 2.5) the QCD matrix elements factorise into the Born-level matrix elements and a factor that takes into account the colour flow due to the soft-gluon emission. In the limit in which two partons become collinear, the collinear pair can be replaced by a single hard parton and the QCD matrix elements factorise into the Born-level matrix elements and the Altarelli-Parisi splitting functions. It follows that the leading higher-order real corrections are proportional to the LO amplitude and, therefore, it is possible to factorise the contributions from real QCD emission with respect to the process-dependent virtual amplitude.

The complete factorisation of the cross section requires kinematics to factorise as well. An essential property is phase-space factorisation, which ensures that the *integrated* matrix element squared, i.e. the cross section, is factorisable. In the Sudakov limit $x \to 0$, the phase-space Φ is factorisable if it can be approximated to [44]

$$d\Phi(\{p_i\};q) \sim d\Phi(\{p_i\}) \, \frac{d^4q}{(2\pi)^3}\delta_+(q^2) \, u(\{p_i\};q)\,, \tag{2.32}$$

where $d\Phi(\{p_i\})$ is the Born-level phase space and q is the momentum of an additional final-state parton that is soft or collinear to the momenta $\{p_i\}$. The Sudakov weight $u(\{p_i\};q)$ depends on the observable, with its LO kinematics, and on the momentum q. In case of multiple radiation of partons with momenta q_i, the phase space is factorisable if all the Sudakov weights are uncorrelated. The factorisation formula follows from the iteration of Eq. (2.32) and each Sudakov weight depends on a single infrared momentum q_i. This iterative structure is typically not valid in the kinematical space where the cross sections are defined, but in some conjugate space. Depending on the specific observable under consideration, there will be certain soft and collinear degrees of freedom in the phase space for single emission that are correlated between different emissions. These degrees of freedom are convoluted via energy and momentum conservation constraints that define the measured kinematical variables, like the invariant mass and the transverse momentum of an observed final-state system.

These convolutions, non-factorisable in the direct space, are turned into simple products by applying properly defined transformations. In the case of threshold resummation, the constraint of energy conservation is factorised in Mellin space [33]. In the case of transverse-momentum resummation, the constraint of transverse-momentum conservation is factorised by performing a Fourier transform to the impact-parameter space [26, 27]. Explicitly, for multiple emissions of m real partons of four-momenta q_i we can define transferred energies z_i and transverse momenta $\mathbf{q}_{i\perp}$. If the total transferred energy z is measured, which is the case of threshold resummation, the Mellin transform of the conservation constraint from z-space to N-space is

$$\int dz\, z^{N-1}\, \delta(z - z_1 z_2 \dots z_m) = \prod_{i=1}^{m} z_i^{N-1} \, . \tag{2.33}$$

The r.h.s. of Eq. (2.33) is fully factorised and each z_i^{N-1} factor realises the Mellin transform from z_i-space to N-space. If the total transverse momentum $\mathbf{q}_\perp$ is measured, which is the case of transverse-momentum resummation, the Fourier transform of the conservation constraint from $\mathbf{q}_\perp$-spaces to the space of the impact-parameter $\boldsymbol{b}$ is

$$\int d^2\mathbf{q}_\perp\, e^{-i\mathbf{b}\cdot\mathbf{q}_\perp}\, \delta^{(2)}(\mathbf{q}_\perp - (\mathbf{q}_{1\perp} + \mathbf{q}_{2\perp} \dots + \mathbf{q}_{m\perp})) = \prod_{i=1}^{m} e^{-i\mathbf{b}\cdot\mathbf{q}_{i\perp}} \, . \tag{2.34}$$

The r.h.s. of Eq. (2.34) is fully factorised and each $e^{-i\mathbf{b}\cdot\mathbf{q}_{i\perp}}$ factor realises the Fourier transform from $\mathbf{q}_{i\perp}$-space to $\mathbf{b}$-space.

2.3 Soft-gluon effects

The correct way to deal with large logarithmic terms of collinear origin is to resum them into the definition of the parton densities, according to the solutions of Eq. (2.20). The factorisation scale μ_F^2 should then be chosen of the order of the hard scale Q^2, so that the residual logarithms $\ln(Q^2/\mu_F^2)$ do not themselves spoil the convergency of the perturbative series. The same holds for registered partons in the final state. The fragmentation functions follow a similar evolution equation as the parton densities, and they should be evaluated at a (possibly) different scale $\mu_f^2 \sim Q^2$.

According to the factorisation theorem any infrared-safe observable defined for hadronic events can be written as a convolution of Parton Distribution Functions (PDFs) and parton coefficient functions, commonly referred to as partonic cross sections. As discussed in the introduction, the coefficient functions can themselves be affected by other large logarithmic terms, if more than one hard scale is present. The issue is very similar to the one discussed in the previous section, where the hadron scale $Q_0^2 \sim M_h^2$ is replaced by another scale $Q_0^2 \gg M_h^2$, which is however soft with respect to Q^2.

The question of whether a valid perturbative approach can be defined, and specifically if and how the large logarithms can be resummed, is not limited to the AP evolution equation for the PDFs, but it must be extended to more generic processes sensitive to soft-gluon emission. We first consider a simple example: a jet produced at small invariant mass from e^+e^- collisions. It is a two-scale problem, where the centre-of-mass energy $Q^2 > 0$ is large with respect to the jet mass m_J^2. At the leading-order in α_S the jet has no structure and the final-state system consists of two massless partons. One of them can be identified with the jet of measured mass, with LO momentum p and energy $E_p \sim Q/2$. The other will be the recoiling jet. At the NLO, the real correction is due the emission of an on-shell gluon of momentum q and energy ω, radiated by one of the final-state partons at an angle θ. The differential probability in the collinear limit is

$$\frac{\alpha_S}{\pi}\, dw_r^{(1)} = 2\, C_a \frac{\alpha_S}{\pi} \frac{d\omega}{\omega} \frac{d\theta^2}{\theta^2}\,. \tag{2.35}$$

The physics of the emission is the same as Eq. (2.7), where C_a is the (squared) charge of the emitter, $d\omega/\omega$ is the bremsstrahlung spectrum and $d\theta^2/\theta^2$ is the collinear-splitting spectrum. The factor 2 takes into account both the QCD emitters. Only the soft-collinear *eikonal* term of the AP splitting probability has been considered and the non-soft collinear terms have been neglected (see Section 2.4). The kinematics for the emission is constrained by the m_J^2 scale, which defines the maximal mass squared to be measured for the identified jet. If the LO cross section is $\sigma^{(0)}$, then

$$\sigma = \sigma^{(0)} \left[1 + \frac{\alpha_S}{\pi} \int \left(dw_r^{(1)} + dw_v^{(1)} \right) \Theta(m_J^2 - m_{\text{jet}}^2) + \mathcal{O}(\alpha_S^2) \right], \tag{2.36}$$

where m_{jet} is the jet mass defined at the partonic level. With the NLO real kinematics, the mass of the jet is given by $2p \cdot q = 2\omega E_p(1 - \cos\theta) \sim \omega E_p \theta^2$, so that

$$w_r^{(1)} = 2\, C_a \int_0^{E_p} \frac{d\omega}{\omega} \int_0^1 \frac{d\theta^2}{\theta^2}\, \Theta(m_J^2 - \omega E_p \theta^2)\,. \tag{2.37}$$

There are two possible sources of infrared divergencies. One is due to the soft limit $\omega \to 0$, or $z \to 1$. The other is the collinear singularity in the $\theta \to 0$, or $q_{\mathrm{T}} \to 0$, limit. The virtual correction is due to the self-energy diagram, where a virtual gluon is emitted and reabsorbed by the same parton. The differential probability is the same of the real one in Eq. (2.35) and opposite in sign, $dw_v^{(1)} = -dw_r^{(1)}$. The opposite sign is due to unitarity and it ensures the cancellation of the singularities, according to the KLN theorem. The real radiation enhances the cross section, while the virtual radiation suppresses it. The kinematics of the two contributions is not the same. The virtual diagram has the same kinematics as the LO, with a zero jet mass, so that the Heaviside-theta function of Eq. (2.37) is replaced by 1. The sum of the real and the virtual probability gives

$$w_r^{(1)} + w_v^{(1)} = -2\, C_a \int_0^{E_p} \frac{d\omega}{\omega} \int_0^1 \frac{d\theta^2}{\theta^2}\, \Theta(\omega E_p \theta^2 - m_J^2). \tag{2.38}$$

The soft region $\omega < m_J^2/E_p$ and the collinear region $\theta^2 < m_J^2/(\omega E_p)$ cancel out in the difference. The total cross section up to the NLO corrections is

$$\sigma = \sigma^{(0)} \left[1 - \frac{\alpha_{\mathrm{S}}}{\pi} C_a \ln^2 \frac{Q^2}{4m_J^2} + \mathcal{O}(\alpha_{\mathrm{S}}^2) \right]. \tag{2.39}$$

The two-scale logarithm $\ln(Q^2/m_J^2)$ can be large in the regime $m_J^2 \ll Q^2$, because of a strong kinematical suppression of the real radiation, so that $\alpha_{\mathrm{S}} \ln(Q^2/m_J^2) \sim 1$. The cross section can even become negative! The cancellation of the infrared poles is ensured by unitarity, but it is highly unbalanced, at any order of α_{S} in perturbation theory.

A similar result holds for the one-particle inclusive cross section, which is the topic of Chapter 4. In the following we still consider e^+e^- collisions, for simplicity. The cross section does not factorise anymore in the phase-space where it is defined and a Mellin transform is required. The Sudakov parameter is $x = 2p \cdot Q/Q^2 < 1$, where Q is the total momentum available in the centre-of-mass frame and p is the momentum of the registered final-state parton. Under a proper LO normalisation $\sigma^{(0)}$, which does not include the delta distribution of x, the cross section can be written as

$$d\sigma(x) = \sigma^{(0)} \left[\delta(1-x) + \frac{\alpha_{\mathrm{S}}}{\pi} \left(w_r^{(1)}(x) + w_v^{(1)}(x) \right) + \mathcal{O}(\alpha_{\mathrm{S}}^2) \right], \tag{2.40}$$

where $w_v^{(1)}$ and $w_r^{(1)}$ are the probabilities of virtual and real emission, respectively. The virtual emission probability is equal to

$$w_v^{(1)}(x) = -C_a \int_0^{E_p} \frac{d\omega}{\omega} \int_0^1 \frac{d\theta^2}{\theta^2}\, \delta(1-x)\, \Theta(q_{\mathrm{T}}^2 - \mu_f^2). \tag{2.41}$$

To draw a full analogy with the previous section, the energy fraction is now $z = 1 - \omega/E_p$ and the transverse momentum in the collinear limit is $q_{\mathrm{T}} \sim \omega\theta$. The collinear limit is actually protected by the factorisation scale μ_f^2: the final-state parton is not the physical state to be observed. The fragmentation function describes the evolution of the free parton into a hadronic final state and it comprises the radiation with small transverse momentum $q_{\mathrm{T}}^2 < \mu_f^2$. The real emission probability is equal to

$$w_r^{(1)}(x) = +C_a \int_0^{E_p} \frac{d\omega}{\omega} \int_0^1 \frac{d\theta^2}{\theta^2}\, \delta(1 - x - \omega/E_p)\, \Theta(q_{\mathrm{T}}^2 - \mu_f^2)\,. \tag{2.42}$$

The LO kinematical constraint $\delta(1 - x)$ of Eq. (2.40), valid for the virtual corrections (2.41) as well, is due to the zero-mass condition on the recoiling parton of momentum $p_r = Q - p$. When a real gluon with momentum q is radiated, the recoiling parton has momentum $p_r = Q - p - q$ instead. Now the zero-mass condition leads to $\delta(1 - x - 2p_r \cdot q/Q^2)$. If the gluon is emitted with energy ω at an angle θ with respect to p, then $2p_r \cdot q \sim 2\omega E_p(1 + cos\theta) \sim 4\omega E_p$. The delta function in Eq. (2.42) follows from the relation $E_p = \sqrt{Q^2}/2$, valid in the soft limit. The distributions of the x variable, which appear in Eq. (2.40) but not in Eq. (2.36), are due to the completely exclusive phase-space of the registered parton. In the previous example, the production of a light jet, the mass of the jet is instead integrated up to the upper boundary m_J. By performing the same integration on Eq. (2.40), i.e. the integral of $d(2p \cdot q)$ over the range $(0, m_J^2)$, the x parameter is replaced by the m_J/Q ratio, which effectively becomes the new Sudakov parameter. The LO constraint is then turned into a constant and the NLO constraint gives rise to the Heaviside theta function of Eq. (2.37).

The Mellin moment of the total emission probability with respect to x is completely factorised with respect to the LO cross section. In terms of the energy fraction z and the transverse momentum $q_{\mathrm{T}} \sim \theta(1 - z)Q$,

$$w_{r,N}^{(1)} + w_{v,N}^{(1)} = C_a \int_0^1 \frac{dz}{1 - z} \int_{\mu_f^2}^{(1-z)^2 Q^2} \frac{dq_{\mathrm{T}}^2}{q_{\mathrm{T}}^2} \left(z^{N-1} - 1\right), \tag{2.43}$$

so that the total cross section in Mellin space, up to the NLO, reads

$$d\sigma_N = \sigma^{(0)} \left[1 + \frac{\alpha_{\mathrm{S}}}{\pi} C_a \int_0^1 dz\, \frac{z^{N-1} - 1}{1 - z} \left(\ln(1 - z)^2 + \ln \frac{Q^2}{\mu_f^2} \right) + \mathcal{O}(\alpha_{\mathrm{S}}^2) \right]. \tag{2.44}$$

The expression in the r.h.s. of Eq. (2.44) is finite in the soft limit $z \to 1$, because the pole is regularised by the numerator $(z^{N-1} - 1)$, which is due to the mismatch of the real kinematics ($x = z$) and the virtual kinematics ($x = 1$). The N-moments in Eq. (2.44) were computed in Ref. [35]. It was shown that, in the large-N limit ($x \to 1$ limit), the Mellin transform can be calculated via the approximation

$$z^{N-1} \sim \Theta(z - (1 - N_0/N)) \,, \tag{2.45}$$

where N_0 is a numerical coefficient. The choice $N_0 = e^{-\gamma_E}$ ($\gamma_E = 0.5772\ldots$ is the Euler number) allows us to control the large logarithmic terms up to NLL accuracy. With this simple prescription it can be easily obtained that the plus distributions $[\ln(1-z)/(1-z)]_+$ and $[1/(1-z)]_+$ correspond to double- and single-logarithmic terms $\alpha_S \ln^2(N)$ and $\alpha_S \ln(N)$, respectively: one logarithm is associated to the soft limit of the Bremsstrahlung spectrum $dz/(1-z)$, the other logarithm is due to the transverse-momentum spectrum in the collinear limit $q_T^2 \sim (1-z)^2 Q^2$. These terms are both large in the large-N limit and they should be resummed to all orders.

These simple calculations show that cross sections of observables that are close to the inclusive boundaries of the phase-space feature up to two powers of large logarithms for one power of α_S. Higher-order terms, formally suppressed with respect to lower-order terms, are made so large that they are not negligible anymore. The fixed-order calculation is not adequate in order to extract phenomenologically relevant results and an improved perturbative treatment is required. We note that process-dependent kinematical details, such as the hard integration boundary on the longitudinal-momentum fraction and the large-angle boundary on the transverse-momentum, affects the structure of the large logarithmic corrections. The enhanced logarithmic terms are not universal and in principle they must be evaluated process-by-process. In practice it is possible to develop resummation strategies valid for an entire class of processes that share similar kinematical features, so that one formula can account for the resummation of several cross sections. The details on how the logarithms are better organised and exponentiated differ from process to process (or from class of processes to class of processes) and may subject to arbitrary choices. In this Chapter we focus on the fundamental techniques that make the all-order calculation of large logarithmic contributions possible. The factorisation and resummation is better understood in the simpler case of the abelian $U(1)$ gauge theory of QED, where the force carrier, the photon, is electrically neutral. The same approach is then extended to correctly include the non-abelian dynamics of gluon radiation in QCD.

2.4 Soft emission in QED

In the soft limit the essential feature of QED is the complete factorisation of the infrared singular contributions with respect to the regular terms. The divergent terms can be completely expressed in terms of the momenta of the external particles and are independent of the hard interaction. This makes the all-order treatment of the large logarithmic terms possible.

We start with the definition of the eikonal current. Let p be the momentum of an external (outgoing) fermion line in a QED process. The emission of a real photon of momentum q and helicity λ from this on-shell fermion of mass m is described by the matrix element

$$\bar{u}(p)(ie\gamma^{\mu})\frac{-i(\not{p}+\not{q}+m)}{(p+q)^2-m^2}\epsilon_{\mu}^{\lambda}(q)\,, \tag{2.46}$$

The spinor $\bar{u}(p)$ is due to the on-shell fermion line and is independent of the emission. The four-vector ϵ_{μ} is the polarisation vector of the emitted photon. The other terms are due to the fermion-photon interaction vertex and to the virtual fermion propagator. From the on-shellness of the external photon and the external fermion, the denominator of Eq. (2.46) reduces to $2p\cdot q$. The spin structure can be simplified with the commutation property of the Dirac γ matrices and by taking advantage of the on-shell Dirac equation $\bar{u}(p)(\not{p}-m)=0$. In the soft limit $q\to 0$ the $\not{q}$ term is negligible and the $\bar{u}(p)$ is simply multiplied by the *eikonal factor*

$$e\frac{p^{\mu}}{p\cdot q}\epsilon_{\mu}^{\lambda}(q)\,. \tag{2.47}$$

The same result is found for the emission of a real photon from an ingoing external fermion line, with the replacement $e\to -e$. The emission of a real soft photon from a diagram with m external fermion lines can be expressed in terms of the sum of the eikonal vectors

$$J^{\mu}(q)=\sum_{i=1}^{m}\frac{e_i\,p_i^{\mu}}{p\cdot q}\,. \tag{2.48}$$

The charge e_i in Eq. (2.48) is equal to the electric charge of lepton i in unit of the electron charge e, if the fermion line is outgoing and to its opposite if the fermion line is ingoing. With this choice, the conservation of the total electric current reads

$$q_{\mu}J^{\mu}(q)=\sum_{i=1}^{m}e_i=0\,. \tag{2.49}$$

If the renormalised transition amplitude for the diagram with m external fermions (and no soft photons) is $\mathcal{M}_0$, the real amplitude $\mathcal{M}_1$ for the emission of a final-state soft photon is

$$\mathcal{M}_1^\lambda(p_1,\ldots,p_m;q) \sim e\,\epsilon_\mu^\lambda(q) J^\mu(q)\,\mathcal{M}_0(p_1,\ldots,p_m)\,. \tag{2.50}$$

If the $\mathcal{M}_0$ amplitude is finite (free of infrared singularities), the $\mathcal{M}_1$ amplitude is not, due to the divergent structure of J^μ in the soft (or collinear) limit $p\cdot q \to 0$. The singular contribution is then completely factorised from the original amplitude $\mathcal{M}_0$ and it is expressed in terms of a vector, J^μ, that is independent of the details of the hard (non-soft) process. The physical interpretation is that large-wavelength radiation can not resolve the internal structure of short-distance interactions.

The single-photon emission probability $dw_r^{(1)}$ is defined, at the amplitude-squared level, as

$$|\mathcal{M}_1(p_1,\ldots,p_m;q)|^2 \sim \frac{\alpha}{\pi}\,dw_r^{(1)}(q)\,|\mathcal{M}_0(p_1,\ldots,p_m)|^2\,, \tag{2.51}$$

where $\alpha = e^2/(4\pi)$ is the fine-structure constant. The photon phase-space

$$[dq] = \frac{d^4q}{(2\pi)^3}\delta_+(q^2) = \frac{d^3q}{(2\pi)^3\,2\omega}\,, \tag{2.52}$$

and the sum over the photon polarisations,

$$\sum_\lambda [\epsilon_\mu^\lambda(q)]^* \epsilon_\nu^\lambda(q) = d^{\mu\nu}(q)\,, \tag{2.53}$$

are included in $dw_r^{(1)}$. The photon energy is ω. After multiplying the expression in Eq. (2.50) by its complex conjugate, and by factorising the amplitude-squared of the short-distance process, one gets

$$dw_r^{(1)}(q) = [dq](2\pi)^2 J^\mu(q) d_{\mu\nu}(q) J^\nu(q) = \frac{d^3q}{4\pi\omega_q}\left(-J_\mu(q)J^\mu(q)\right)\,. \tag{2.54}$$

The photon polarisation tensor $d^{\mu\nu}$ contains longitudinal terms that are gauge dependent, but the probability is independent of such terms and hence gauge invariant. Indeed any longitudinal term, proportional to the momentum q^μ, vanishes as a consequence of charge conservation in Eq. (2.49). The only term to give a non-zero contribution is the tensor $-g^{\mu\nu}$.

The explicit expression of $dw_r^{(1)}$ is

$$dw_r^{(1)}(q) = \frac{d^3q}{4\pi\omega}\sum_{i,j\neq i} e_i e_j \frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q}\,. \tag{2.55}$$

For simplicity all the lepton are considered massless, so that $p_i^2 = 0$. In addition, all the $p_i \cdot q$ are vanishing both in the soft and in the collinear emission limit. The expression in Eq. (2.55) contains double poles, which in turn are known to give double-logarithmic terms, once the virtual contribution is subtracted. From unitarity, the virtual emission probability is equal to the real one but opposite in sign. The different kinematics (loop integral versus phase-space integral) leads to the logarithms in the total emission probability. If a lepton is massive, the collinear limit is not singular and there is only a soft pole, hence a single non-collinear logarithm. Since the eikonal approximation has been used in the first place, where non-leading terms in the soft limit are neglected, all the single-logarithmic terms can be neglected as well.

In the leading soft-collinear approximation, the photon has to be collinear to one fermion. Considering Eq. (2.55), if the photon is emitted from lepton i within an angle θ_{iq} which is much smaller than the angles θ_{ij} between the lepton momenta, then $p_i \cdot q \sim (E_i \omega_q)\theta_{iq}^2/2$ and $p_j \cdot q \sim (\omega_q/E_i) p_i \cdot p_j$. For each momentum p_i to which the photon momentum q is collinear, Eq. (2.55) is independent of any other momentum p_j. The sum over j is a sum over the colour charges and, after charge conservation (2.49), it simply gives the charge $-e_i$. The result is a sum over independent emitters,

$$\frac{\alpha}{\pi} dw_r^{(1)}(q) = \frac{\alpha}{\pi} \sum_{i=1}^{m} e_i^2 \frac{d\omega}{\omega} \frac{d\theta_{iq}^2}{\theta_{iq}^2} \Theta(Q-\omega)\Theta(\theta_0 - \theta_{iq}) \,. \tag{2.56}$$

The large-angle radiation cancels due to quantum-interference destructive effects. Photons radiated at large angles can not resolve the single charges of the hard process and 'see' a zero net charge. On the contrary, the soft-collinear photon can only be sensitive to the charge of its emitter. This effect is known as QED coherence. The Heaviside theta functions in Eq. (2.56) remind us that the expression is valid in the soft limit at scales that are controlled by the hard scale Q and in the collinear limit where the emission angle is smaller than $\theta_0 \sim \min_{ij} \theta_{ij}$. It should be noted that the spectrum of each emitter is analogous to the single-emission QCD spectrum of Eq. (2.35), with the obvious replacements of the coupling and the fermion charge. The reason is that only the eikonal term of the Altarelli-Parisi splitting probability is present in Eq. (2.35). The eikonal approximation in QCD is further discussed in Section 2.5.

In the case of multiple soft-photon emission the picture is the same, just iterated. After the emission of the first photon of momentum q_1, the momentum and the electric charge of the external leptons are unchanged. The

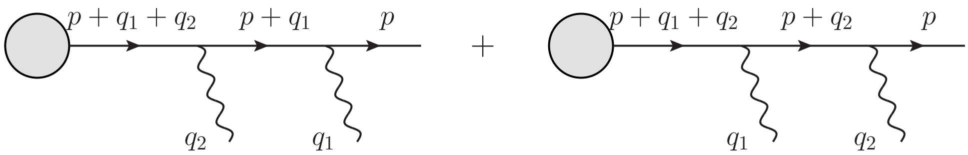

Figure 2.6: Emission of two photons from a fermion line.

probability for the emission of n soft photon is given by the product of the single emission probability $dw_r^{(1)}$ for each photon, defined in Eq. (2.55), multiplied by a $1/n!$ symmetry factor for n identical bosons in the final-state,

$$\left(\frac{\alpha}{\pi}\right)^n dw_r^{(n)}(q_1,\ldots,q_n) \sim \frac{1}{n!}\left(\frac{\alpha}{\pi}\right)^n \prod_{i=1}^{n} dw_r^{(1)}(q_i)\,. \tag{2.57}$$

Eq. (2.57) can be proved starting from the matrix elements. If the photons are radiated from different leptons, the total contribution is the simple product of the eikonal coefficients of each line. In case of multiple emission from the same lepton, the total contribution is equal to the sum of all the possible ordered diagrams. For example, the double emission eikonal factor to multiply the $\mathcal{M}_0$ matrix-element is

$$e^2 \frac{p^{\mu_1} p^{\mu_2}}{p\cdot(q_1+q_2)+q_1\cdot q_2}\left(\frac{1}{p\cdot q_1}+\frac{1}{p\cdot q_2}\right)\epsilon_{\lambda_1,\mu_1}(q_1)\,\epsilon_{\lambda_2,\mu_2}(q_2)\,. \tag{2.58}$$

The denominator in the first fraction of Eq (2.58) is due to the lepton propagator before any emission. The fractions in curly brackets are due to the propagator between the two emission vertices, one for each contributing diagram (Figure 2.58). By neglecting the subleading term $q_1 \cdot q_2$ in the first denominator and by combining the two fractions in the curly brackets, the result is the simple product of two independent eikonal factors. The same argument can be generalised to the emission of n soft gluons and the corresponding matrix element $\mathcal{M}_n$ is

$$\mathcal{M}_n^{\lambda_1\ldots\lambda_n}(p_1,\ldots,p_m;q_1,\ldots,q_n) \sim e^n \mathcal{M}_0(p_1,\ldots,p_m)\prod_{i=1}^{n} J^{\mu_i}(q_i)\epsilon_{\mu_i}^{\lambda_i}(q_i)\,. \tag{2.59}$$

The square of (2.59), summed over the polarisation λ_i and multiplied by the phase-space factor $[dq_i]$ of each soft-photon, straightforwardly leads to the multiple emission probability dw_n defined in Eq. (2.57).

The all-order real emission probability is equal to the sum of all the multiple emission probabilities $dw_r^{(n)}$, integrated over the phase space $[dq_i]$ of each emitted photon, with a weight $u(q_i)$ fixed by the kinematics. From the multiplicative form of (2.57),

$$1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\alpha}{\pi}\right)^n\prod_{i=1}^{n}\int dw_r^{(1)}(q_i)u(q_i)=\exp\left\{\frac{\alpha}{\pi}\int dw_r^{(1)}(q)u(q)\right\}\,. \quad (2.60)$$

We stress that Eq. (2.60) is valid if and only if the phase space of multiple emissions simply factorises in the product of phase spaces for single emission, according to the iterated version of Eq. (2.32) (see Section 2.2).

From unitarity, the virtual differential emission probability must be equal to the real one and opposite in sign. The virtual emission has however the same kinematics as the LO process and no further constraint, so that the total emission probability w is given by the replacement $u(q)\to u(q)-1$,

$$w=\exp\left\{\frac{\alpha}{\pi}\int dw_r^{(1)}(q)\left[u(q)-1\right]\right\}\,. \quad (2.61)$$

Eq. (2.61) takes into account the all-order soft-photon emission and, with the single emission probability $dw_r^{(1)}$ of Eq. (2.56), it resums the leading logarithmic terms to all orders in α. Note that the normalisation of w in Eq. (2.61) is the correct one: the total real soft emission, without kinematical constraints, completely cancels the virtual soft emission ($w\to 1$).

2.5 Factorisation in QCD

If we consider a non-abelian theory like QCD, the gluon self-interactions play a role. The diagrams for the emission of a soft gluon from a hard gluon contribute to the total probability and they break the factorisation of the singular terms. In addition, a gluon always carries a colour charge, no matter how soft it is. The colour flow is more involved than the electric current of QED, which flows only through fermion lines. Now the colour charge is a matrix in colour space, acting on the the colour vector associated to the hard matrix element. The extension of the eikonal approximation to QCD requires some ingenuity. It can be shown that the factorisation holds for the gluon vertex under a proper gauge choice and that any emitter behaves in the same way in the soft-collinear approximation, being it a fermion or a gluon. The multiple emission case is even more complicated, because of the non-commutative colour charges.

The eikonal factor for a fermion in QCD can be derived via the same steps followed in the previous Section, with the photon-interaction vertex replaced by the gluon-interaction vertex. If a soft gluon of momentum q, helicity λ and colour a is emitted from an outgoing fermion of momentum p and colour i, the real emission amplitude is equal to

$$[\mathcal{M}_1^{\lambda}(p;q)]_i^a = [\mathcal{M}_0(p)]_j \, g_{\mathrm{S}} \, t^a_{ji} \frac{p^\mu}{p\cdot q} \, \epsilon^\lambda_\mu(q) \, . \tag{2.62}$$

For simplicity the dependence on the colours and momenta of the hard process, apart from the emitter, is left implicit. The colour tensors t^a_{ji} are the generators of $SU(N_c)$ in the fundamental representation, of dimension N_c. In QCD, the indices i,j run from 1 to $N_c = 3$ and the index a from 1 to $N_c^2 - 1 = 8$. The same amplitude describes the emission from an incoming antifermion. For an incoming fermion and an outgoing antifermion Eq. (2.62) is valid with the replacement $t^a_{ji} \to -(t^a_{ji})^t = -t^a_{ij}$.

From Eq. (2.62) it is clear that the eikonal current is not a simple Lorentz vector like in QED, but an operator in colour space. The emission of the same soft gluon from a hard gluon of momentum p, helicity λ_p and colour b is described, in the $q \to 0$ limit, by the amplitude

$$[\mathcal{M}_1^{\lambda_p\lambda}(p;q)]^{ab} = (\mathcal{M}_0^\sigma)_c \left[-g_{\mathrm{S}} f^{abc} V^{\mu\nu\rho}(-p,0,p)\right] \frac{i d_{\rho\sigma}(p)}{2p\cdot q} \epsilon^{\lambda_p}_\mu(p) \epsilon^\lambda_\nu(q) \, , \tag{2.63}$$

where the three-gluon vertex is

$$V_{\mu\nu\rho}(-p,0,p) = -g_{\mu\nu} p_\rho - g_{\nu\rho} p_\mu + 2 g_{\rho\mu} p_\nu \, , \tag{2.64}$$

and the gluon polarisation tensor can be decomposed in terms of the polarisation vectors as in Eq. (2.53). The colour tensors f^{abc} are the generators of $SU(N_c)$ in the adjoint representation, of dimension $N_c^2 - 1$. The four-gluon vertex, leading to subleading corrections, has been neglected. The first two terms in the r.h.s. of Eq. (2.64) produce a contribution proportional to $p\cdot\epsilon(p)\epsilon^\mu(p)$, while the third term results in $\epsilon^2(p)p^\mu$. It follows that the matrix element of Eq. (2.63) depends on the spin of the hard emitter. To overcome this problem we can choose a physical gauge, where the gluon longitudinal polarisations vanish, $p\cdot\epsilon(p) = 0$, and the normalisation is $\epsilon^2(p) = 1$. With this gauge choice only the last term in Eq. (2.64) survives and

$$[\mathcal{M}_1^{\lambda_p\lambda}(p;q)]^{ab} = [\mathcal{M}_0^\sigma(p)\epsilon^{\lambda_p}_\sigma(p)]_c \, g_{\mathrm{S}} \, i f^{abc} \frac{p^\nu}{p\cdot q} \, \epsilon^\lambda_\mu(q) \, . \tag{2.65}$$

The term in square bracket in the r.h.s. of Eq. (2.65) is the Born-level amplitude $[\mathcal{M}_0]_c$.

Comparing Eq. (2.62) with Eq. (2.65) we see that the eikonal factor for a gluon emitter is analogous to the eikonal factor for a fermion emitter. The non-abelian colour charge t^a_{ji} for the fermion is simply replaced by the non-abelian colour charge if^{abc} for the gluon. In either case the factorisation is not a scalar factorisation, but holds at a tensor level. It is possible to summarise these results in a single expression, where the factorisation can be expressed in terms of colour vectors and colour matrices, by making use of the colour space notation [66, 67]. The colour index c of a parton, being it a fermion or a gluon, is represented by a vector in an abstract colour space. The possible quantum states of m partons are represented by a base of vectors $|c_1 \ldots c_m\rangle$ in the Fock space and the amplitude of a scattering process between the partons $a_1, \ldots, a_m$ can be thought of as the projection of a vector over this base,

$$\mathcal{M}^{c_1 \ldots c_m}_{a_1 \ldots a_m} = \langle c_1 \ldots c_m | \mathcal{M}_{a_1 \ldots a_m} \rangle \,. \tag{2.66}$$

The radiation from the parton a_i of a gluon of colour c is described by the colour charge matrix $(\mathbf{T}_i)^c$. The colour flow is treated as outgoing, such that

$$\begin{aligned}
(\mathbf{T}_i)^c_{ij} &= t^a_{ij}\,, && \text{if } a_i \text{ is an outgoing } q \text{ or an ingoing } \bar{q}\,,\\
(\mathbf{T}_i)^c_{ij} &= -t^a_{ji}\,, && \text{if } a_i \text{ is an ingoing } q \text{ or an outgoing } \bar{q}\,,\\
(\mathbf{T}_i)^c_{bc} &= if_{cab}\,, && \text{if } a_i \text{ is a gluon}\,.
\end{aligned} \tag{2.67}$$

The sign choice is a natural extension of the negative electric charge assigned to an ingoing fermion. With this choice $|M\rangle$ is a colour singlet and the conservation of the total colour charge reads

$$\sum_{i=1}^{m} \mathbf{T}_i | \mathcal{M}_{a_1 \ldots a_m} \rangle \;= 0\,. \tag{2.68}$$

The exchange of a gluon between two partons i and j corresponds to the product

$$\mathbf{T}_i \cdot \mathbf{T}_j = \sum_c (\mathbf{T}_i)^c (\mathbf{T}_j)^c\,, \tag{2.69}$$

The colour algebra for this product gives

$$\mathbf{T}_i^2 = C_{a_i}\,, \qquad \mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i\,, \tag{2.70}$$

where C_{a_i} is the Casimir factor of $SU(N_c)$ in the fundamental or adjoint representation. If a is a gluon then $C_a = C_A = N_c$, if a is a quark or an antiquark then $C_a = C_F = (N_c^2 - 1)/(2N_c)$. The Casimir is a c-number and corresponds to the colour structure of a self-interaction diagram, where the gluon is emitted and reabsorbed by the same parton. It is essentially a multiple of the identity operator in colour space. Conversely, the quadratic operator $\mathbf{T}_i \cdot \mathbf{T}_j$ introduces non-trivial colour correlations if $i \neq j$, related to the colour flow from parton i to parton j. It is now possible to summarise the results of Eq. (2.62) and Eq. (2.65) in one expression. In colour space, the amplitude for the single-emission of a soft gluon is

$$|\mathcal{M}_1^\lambda(p_1, \ldots, p_m; q)\rangle = g_\mathrm{S}\, \epsilon_\mu^\lambda(q) \boldsymbol{J}^\mu(q) |\mathcal{M}_0(p_1, \ldots, p_m)\rangle \,, \tag{2.71}$$

where the total eikonal operator $\boldsymbol{J}^\mu(q)$ is equal to the sum

$$\boldsymbol{J}^\mu(q) = \sum_{i=1}^{m} \mathbf{T}_i \frac{p_i^\mu}{p_i \cdot q} \,. \tag{2.72}$$

Eqs. (2.72) and (2.71) are equivalent to Eq. (2.48) and (2.50) for QED respectively, if the scalar factorisation is replaced by the general factorisation in colour space. The importance of dealing with operators, instead of scalars, is manifest when working with probability. The squared amplitude, summed over the colours of the external partons, is now defined as

$$\sum_{c_1 \ldots c_m} |\mathcal{M}_{a_1 \ldots a_m}^{c_1 \ldots c_m}|^2 = \langle \mathcal{M}_{a_1 \ldots a_m} | \mathcal{M}_{a_1 \ldots a_m} \rangle \,. \tag{2.73}$$

The single-emission probability, defined as in Eq. (2.51), is then

$$\frac{\alpha_\mathrm{S}}{\pi} dw_r^{(1)}(q) = \frac{\alpha_\mathrm{S}}{\pi} \frac{d^3 q}{4\pi\omega} \frac{\langle \mathcal{M}_0(p) | \left(-\boldsymbol{J}_\mu(q) \boldsymbol{J}^\mu(q) \right) | \mathcal{M}_0 \rangle}{\langle \mathcal{M}_0 | \mathcal{M}_0 \rangle} \,. \tag{2.74}$$

The dependence over the flavours and colours of the hard partons is understood. The same arguments already used for the QED case lead to the expression in Eq. (2.74), where, thanks to gauge invariance, the gluon polarisation tensor effectively contribute as $d^{\mu\nu} = -g^{\mu\nu}$. It should now be clear what a non-scalar factorisation implies. While the QED single-emission probability is truly independent of the detail of the hard-scattering process, the QCD one instead depends on the actual colour flow, described by the $\mathbf{T}_i \cdot \mathbf{T}_j$ operators due to the contraction $\boldsymbol{J}_\mu(q)\boldsymbol{J}^\mu(q)$, and by the colour configuration of the hard-scattering amplitude $\mathcal{M}_0$. The numerator in the r.h.s. of Eq. (2.74) is equal to the sum of terms

$$\langle \mathcal{M}_0(p) | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_0 \rangle \,. \tag{2.75}$$

If the operator is a Casimir, i.e. if $i = j$, then the term is proportional to the denominator in the r.h.s. of Eq. (2.74) and the dependence over the hard process cancel out. However, if $i \neq j$, then the colour algebra does

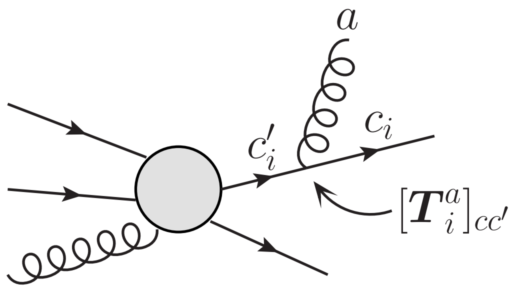

Figure 2.7: Colour operator for the soft-gluon emission vertex.

not factorise. The result depends on the colour dynamics of the interference diagrams contributing to the gluon exchange.

In general the single-emission probability depends on the specific subprocess under consideration. We remind that the simple eikonal approximation accounts for all the terms that are singular in the soft limit. Any term that is singular in the hard-collinear emission limit is neglected. However, soft large-angle radiation is still included. It is reasonable to expect that this non-collinear contributions are responsible for the colour correlations terms $\mathbf{T}_i \cdot \mathbf{T}_j$. On the contrary, a collinear gluon should not be able to see any other charge than the charge of its emitter. At the leading IR approximation, such that only the soft-collinear terms are considered, the emission probability is actually free of any colour correlation. The gluon correlations simply cancel among themselves [68]. This result, known as colour coherence, is the consequence of quantum interference between all the possible single-emission diagrams in $\boldsymbol{J}_\mu \boldsymbol{J}^\mu$ and of gauge invariance, since charge conservation follows from $q^\mu \boldsymbol{J}_\mu = 0$. Following the same reasoning used to derive Eq. (2.56), it can be shown that

$$\frac{\alpha}{\pi}\, dw_r^{(1)}(q) = \frac{\alpha}{\pi} \sum_{i=1}^{m} dw_{r,i}^{(1)}(q)\,, \tag{2.76}$$

where $dw_{r,i}^{(1)}$ is the single-parton-emission probability

$$dw_{r,i}^{(1)}(q) = C_{a_i} \frac{d\omega}{\omega} \frac{d\theta_{iq}^2}{\theta_{iq}^2}\, \Theta(Q-\omega)\, \Theta(\theta_0 - \theta_{iq})\,. \tag{2.77}$$

The angle θ_0 is the minimum of all the scattering angles θ_{ij}. If the real (differential) emission probability is complemented with the virtual one and then integrated over the soft-gluon phase space, within the proper kinematical boundaries, the resulting single-emission probability is free of singularities but it leads to large logarithmic terms. Eq. (2.76) shows that such a probability is free of any colour correlation, so that a true scalar factorisation holds, at the first order in α_S and at leading logarithmic accuracy. The same is not true at higher-orders. The multiple soft-gluon radiation diagrams feature cascade emissions where the history of subsequent splittings determines the colour configuration. After any emission, the initial colour charge of the parent hard parton is shared between the two children partons and the same happens with each subsequent splitting. After n splittings from m initial partons, the colour states of the resulting $n+m$ partons are correlated. Nonetheless, the colour coherent behaviour described by Eq. (2.76) must be followed by each soft emission. The colour of a soft gluon radiated at an angle θ can not be correlated to the colour of gluons radiated at angles $\theta_{ij} \ll \theta$. In a cascade emission, the argument of colour coherence is still valid if the subsequent splittings are more and more collinear. By enforcing angular ordering, the first gluon is radiated at the largest angle and it couples to the hard emitter with a colour charge proportional to the total colour charge of the softer gluons, emitted at smaller angles. This total charge corresponds to the charge of the parent parton and it is therefore independent of the full branching history.

A QED-like factorisation holds for the singular terms of the multiple soft-gluon emission probability, if each hard parton a_i is considered separately,

$$dw_{r,i}^{(n)}(q_1,\dots,q_n) \sim \prod_{j=1}^{n} dw_{r,i}^{(1)}(q_j)\,\Theta(\theta_{i(j-1)} - \theta_{ij})\,, \tag{2.78}$$

and the all-order multiple emission probability is given by the sum over all the independent angular-ordered cascades originated by the hard partons,

$$\left(\frac{\alpha}{\pi}\right)^n dw_r^{(n)}(q_1,\dots,q_n) \sim \left(\frac{\alpha}{\pi}\right)^n \sum_{i=1}^{m} dw_{r,i}^{(n)}(q_1,\dots,q_n)\,. \tag{2.79}$$

With respect to Eq. (2.57), the symmetry factor $1/n!$ is replaced by the angular ordering in Eq. (2.78). This is strictly related to the non-abelian nature of QCD, whose charge operators do not commute, at variance with the scalar charge of QED.

The sum over $n = 1,\dots,\infty$ of the multiple emission probabilities $dw_r^{(n)}$ defined in Eq. (2.79) leads to an exponentiated result. Again, the contribution from each hard parton must be considered on its own. The correct resummed result is given by the product of independently exponentiated $w_{r,i}$

factors,

$$w_r = \prod_{i=1}^{m} w_{r,i}(E_i, \theta_0) \,. \tag{2.80}$$

To preserve QCD coherence at all orders, the concept of angular-ordered partonic cascades must be iterated (Figure 2.8). This leads to a self-similar structure for $w_{r,i}$, implicitly defined by the equation

$$w_{r,i}(E_i, \theta_0) = \exp\left\{ C_{a_i} \frac{\alpha_S}{\pi} \int_0^{E_i} \frac{d\omega}{\omega} \int_0^{\theta_0^2} \frac{d\theta^2}{\theta^2} \, u(q) \, w_{r,q}(\omega, \theta) \right\} . \tag{2.81}$$

The energy E_i is the energy of the hard parton a_i starting the cascade emission and θ_0 is the maximum angle of radiation, which must be smaller then any θ_{ij} angle between the hard emitters. The $w_{r,i}$ probability in the r.h.s. of Eq. (2.81) is the probability of real emission from the child gluon of energy $\omega < E_p$, within an angle $\theta < \theta_0$. Angular ordering is thus guaranteed. The function u is the usual phase-space constraint that defines the kinematics of the real emission. The last step consists of the inclusion of the virtual corrections. In the previous section, the virtual terms were fixed by enforcing the unitarity on the exponentiated result. Along the same lines,

$$w = \prod_{i=1}^{m} w_i(E_i, \theta_0) \,, \tag{2.82}$$

$$w_i(E_i, \theta_0) = \exp\left\{ C_{a_i} \frac{\alpha_S}{\pi} \int_0^{E_i} \frac{d\omega}{\omega} \int_0^{\theta_0^2} \frac{d\theta^2}{\theta^2} \left[u(q) w_q(\omega, \theta) - 1 \right] \right\} . \tag{2.83}$$

Eqs. (2.82) and (2.83) resum to all orders the LL terms $\alpha_S^n L^2 n$ of the soft-gluon emission probability, in a space where the multiple emission kinematics factorises with respect to the LO hard kinematics. The dynamical factorisation is achieved via the eikonal approximation and thanks to the angular-ordered colour-coherent branching algorithm.

This is a first approximation. Ideally, also subleading logarithmic terms should be resummed. At NLO, the NLL terms are the single logarithms, coming from large-angle soft emissions and collinear hard emissions. The latter are simply resummed by the full Altarelli-Parisi splitting function, to be employed in Eq. (2.83) instead of the simple Bremsstrahlung spectrum $d\omega/\omega$. The emission of a parton a of energy ω from a parton b of energy E_b is described by the energy spectrum $dzP_{ba}(z)/z$, where $z = 1 - \omega/E_b$. Large-angle soft radiation is much more complicated, because of colour-interference effects. The leading collinear case is rather simple because the effective charge

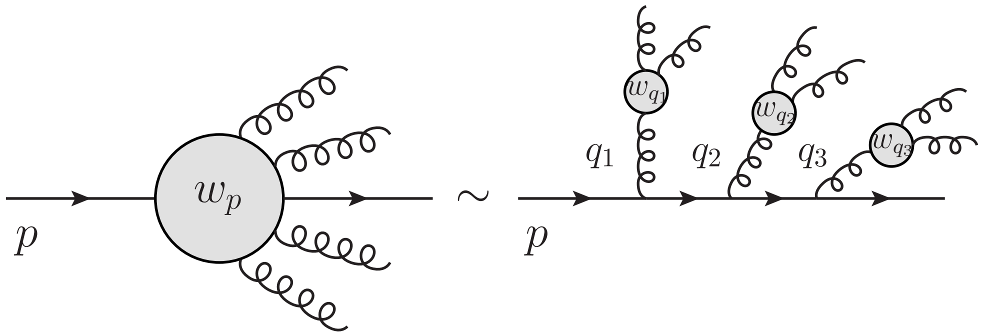

Figure 2.8: Angular-ordered parton shower.

seen by the collinear gluon is the Casimir of the emitter, a unit matrix in colour space. If the angular ordering condition is relaxed, the algebra does not factorise and the multi-gluon emission is expected to be sensitive to the actual colour flow of the hard process. As shown in the next chapters, it is still possible to write resummed results which take advantage of a formal exponentiation. 'Formal' exponentiation means that the argument of the exponential is a colour operator. The formulae are valid in a weak sense, with the meaning that each matrix in colour space must be interpreted as (and replaced by) its action on a specific colour state. The hard amplitude is a vector in colour space, to be decomposed on a colour basis. Each colour operator has its own eigenvalue on each projection. The resummed formulae can be made explicit, in the form of Eq. (3.6), by replacing each colour operator in the argument of the exponent with the proper eigenvalue.

The aim of this Chapter was to illustrate the possible sources of large logarithmic terms and to present the basic concept behind factorisation and exponentiation in abelian and non-abelian gauge theories. To focus on QCD, we have introduced the colour-space formalism and we have showed its importance to the factorisation of the singular infrared contributions. It should also be clear why the same kernel that describes the DGLAP evolution equation of the parton densities and fragmentation functions is to be found in perturbative treatment of soft-gluon effects. The actual resummation formula for an observable of phenomenological interest looks very different to the expression in Eq. (2.83). In addition its structure will strongly depend on the kinematics of the process, i.e. on the functional form of the constraint u.

Chapter 3

Threshold Resummation

In this chapter we focus on the resummation of logarithmic terms that are singular at the partonic threshold. In the near-threshold region, the centre-of-mass energy is just enough to produce the final-state hard particles of the Born-level kinematics and no extra radiation. This is a two-scale problem: if Q^2 is the squared invariant mass of the final-state system and s the squared centre-of-mass energy, then the Sudakov region, the near-threshold region in this case, is approached as $Q^2 \to s$. In this region the real emission is strongly inhibited and the partonic cross section contains large logarithmic terms of argument $1 - Q^2/s$.

According to the factorisation theorem it is possible to decompose an hadronic cross section into the convolution

$$d\sigma_{h_1 h_2 \to F} = \prod_{i=1,2} f_{a_i/h_i}(\mu_F^2) \otimes d\hat{\sigma}_{a_1 a_2 \to F}(\mu_F^2, \mu_f^2)\,, \tag{3.1}$$

where $\sigma_{h_1 h_2 \to F}$ is the hadronic cross section for the production of the final-state system F from $h_1 h_2$ collisions and $\hat{\sigma}_{a_1 a_2 \to a_3 X}$ is the partonic cross section for the scattering between QCD partons of flavours a_1, a_2. The distribution $f_{h/a}$ is the probability, measured at scale μ_F, that parton a from hadron h takes part to the scattering. If the final-state system contains an identified hadron h_3, than Eq. (3.1) is replaced by

$$d\sigma_{h_1 h_2 \to h_3 X} = \prod_{i=1,2} f_{a_i/h_i}(\mu_F^2) \otimes d_{a_3/h_3}(\mu_f^2) \otimes d\hat{\sigma}_{a_1 a_2 \to a_3 X}(\mu_F^2, \mu_f^2)\,, \tag{3.2}$$

where function $d_{h/a}$ is the probability, measured at scale μ_f, that parton a fragments into hadron h. The partonic cross section defined by Eqs. (3.1), (3.2) can be written as

$$d\hat{\sigma} = d\hat{\sigma}^{(\mathrm{reg})} + d\hat{\sigma}^{(\mathrm{sing})} . \tag{3.3}$$

The $d\hat{\sigma}^{(\mathrm{reg})}$ component amounts to the terms which are *regular* in the Sudakov limit. The component $\hat{\sigma}^{(\mathrm{sing})}$ is *singular* in the Sudakov limit: it contains all the Sudakov logarithms, organised order by order in α_S as provided for the standard perturbative approach. In the conjugate space where the cross section admits a full factorisation, the singular component reads

$$d\hat{\sigma}^{(\mathrm{sing})} = d\Phi(\{p_i\}) \, |\mathcal{M}^{(\mathrm{LO})}(\{p_i\})|^2 \, C(\alpha_\mathrm{S}) \, \Sigma(\alpha_\mathrm{S}; L) . \tag{3.4}$$

The factor C contains the hard-virtual terms that are not suppressed in the Sudakov limit but are not logarithmically-enhanced and, therefore, do not need to be resummed. It can be safely expressed as a power series of α_S. The factor Σ realises the *fixed-order* dependence on the Sudakov logarithms L, as a power expansion in α_S. A resummation prescription essentially consists in an improved partonic cross section

$$d\hat{\sigma} = d\hat{\sigma}^{(\mathrm{reg})} + d\hat{\sigma}^{(\mathrm{res})} , \tag{3.5}$$

where $d\hat{\sigma}^{(\mathrm{res})}$ is obtained from $d\hat{\sigma}^{(\mathrm{sing})}$ with the replacement $\Sigma \to \Sigma^{(\mathrm{res})}$. The form factor $\Sigma^{(\mathrm{res})}$ realises the *all-order* exponentiation of the logarithms L. It has the following structure

$$\Sigma^{(\mathrm{res})}(\alpha_\mathrm{S}; L) \sim \exp\{L \, g_1(\alpha_\mathrm{S} L) + g_2(\alpha_\mathrm{S} L) + \alpha_\mathrm{S} \, g_3(\alpha_\mathrm{S} L) + \dots \} . \tag{3.6}$$

The exponent is organised according to the degree of divergence of the various logarithmic terms and not simply as a series of α_S. The function $L \, g_1(\alpha_\mathrm{S} L)$ resums the leading logarithmic (LL) contributions $\alpha_\mathrm{S}^n L^{n+1}$, the function $g_2(\alpha_\mathrm{S} L)$ the next-to-leading logarithmic (NLL) contributions $\alpha_\mathrm{S}^n L^n$ and so forth.

As a case study we choose the Drell-Yan lepton-pair hadroproduction. The resummation for this process was first completed in [33] and [35]. Since the two results are equivalent [36], we refer to the work of Catani and Trentadue, that makes use of the eikonal approximation, the physical gauge, evolution equations and the other techniques introduced in Chapter 1. At the partonic threshold the initial-state partons from the colliding hadrons are the only strong-interacting particles. The colour algebra is simple enough to completely factorise, and there is no need to move to the abstract colour space, introduced in Section 2.5. This allows us to focus on a practical implementation of resummation techniques and to stress the important role played by the kinematics. We then consider the deep-inelastic scattering (DIS) process, where the fragmentation of the scattered parton produces extra enhanced logarithmic terms. The comparison of the two cases, the Drell-Yan and the DIS processes, allow us to define two fundamental ingredients of threshold resummation, the soft-collinear radiative factor and the jet function [69].

3.1 The soft-collinear radiative factor

We consider the production of a lepton pair of momentum Q in the collision of hadrons of momenta P_1 and P_2,

$$h_1(P_1) + h_2(P_2) \to l\bar{l}(Q) + X \,, \tag{3.7}$$

where X is any hadronic final-state recoiling against the leptonic system (Figure 3.1). According to the parton model, at the Born level a quark and an antiquark from the two hadrons produce by annihilation an excited neutral boson, Z or γ^*, that in turn decays into the lepton-pair. The momenta p_1, p_2 of the partons are a fraction of the hadron momenta, $p_i = x_i P_i$ with $x_i \in (0,1)$, with probability described by the PDFs. In the massless approximation, the hadronic and partonic centre-of-mass energies are

$$S = (P_1 + P_2)^2 = 2P_1 \cdot P_2 \,, \qquad s = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = x_1 x_2 S \,. \tag{3.8}$$

The fraction of the centre-of-mass energy carried by the lepton-pair is the quantity of interest when approaching the partonic threshold. It is defined, at the hadronic and the partonic level respectively, as

$$\tau_{\mathrm{H}} = \frac{Q^2}{S} \,, \qquad \tau = \frac{Q^2}{s} = \frac{\tau_{\mathrm{H}}}{x_1 x_2} \,. \tag{3.9}$$

The QCD factorisation theorem ensures that the only mass singularities in this process are due to collinear radiation form the initial-state partons a_1, a_2. As discussed in Chapter 2, these singularities are absorbed by non-perturbative parton densities $f_{a/h}$. The cross section is an infrared-safe quantity, equal to

$$\begin{aligned} Q^2 \frac{d\sigma^{\mathrm{DY}}}{dQ^2}(\tau_{\mathrm{H}}, Q^2) = \sum_{a_1,a_2} \int_0^1 & dx_1 dx_2 \, f_{a_1/h_1}(x_1, \mu_F^2) \, f_{a_2/h_2}(x_2, \mu_F^2) \\ & \times \int_0^1 d\tau \, \delta(\tau_{\mathrm{H}} - x_1 x_2 \tau) \, Q^2 \frac{d\hat{\sigma}^{\mathrm{DY}}_{a_1 a_2}}{dQ^2} \left(\tau, Q^2, \mu_F^2\right) \,, \end{aligned} \tag{3.10}$$

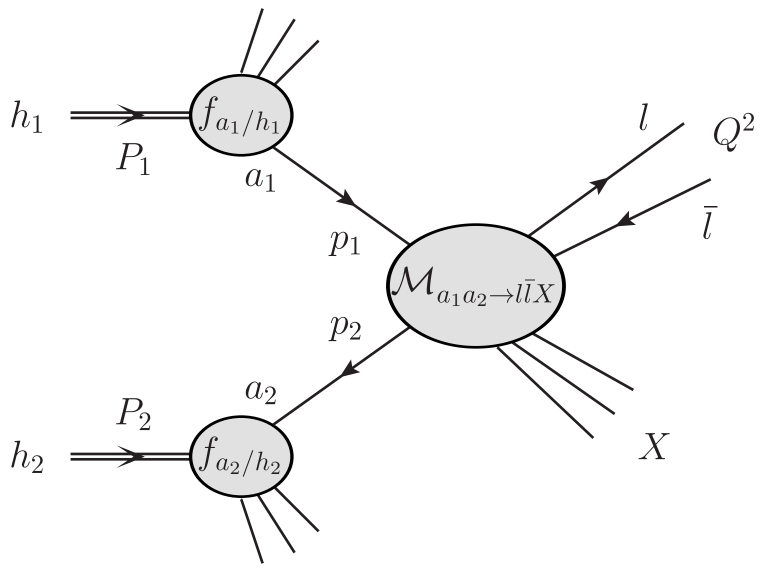

Figure 3.1: Drell-Yan production of a lepton pair.

where μ_F is the factorisation scale.

The QCD radiative corrections are taken into account in the partonic cross section $d\hat{\sigma}$, which is finite and admits a perturbative expansion in $\alpha_S(\mu_R^2)$,

$$Q^2 \frac{d\hat{\sigma}^{\text{DY}}_{a_1 a_2}}{dQ^2}(\tau, Q^2, \mu_F^2) = \sigma_0^{\text{DY}}(\alpha_S(\mu_R^2); Q^2) \left[\delta(1-\tau)\, \delta_{a_1 q}\delta_{a_2 \bar{q}} + \mathcal{O}(\alpha_S(\mu_R^2))\right]. \tag{3.11}$$

The Born cross section σ_0^{DY} fixes the normalisation and μ_R is the renormalisation scale. The lowest-order contribution in the r.h.s. of Eq. (3.11) corresponds to the Born level. At the Born level, the only partonic channel to contribute to the process is quark-antiquark annihilation. With no extra radiation, the lepton invariant mass is forced to be equal to the partonic centre-of-mass energy, which means $\tau = 1$. At higher-orders any flavour channel gives a contribution to the total cross section and the variable τ can assume any value in the range $(\tau_{\text{H}}, 1)$. However, only the terms that are enhanced by soft-gluon effects in the threshold limit $\tau_{\text{H}} \to 1$ are of interest for the purpose of resummation. The energy conservation forces the energy fraction available by the partonic subprocess to the limit $\tau \to 1$ and the longitudinal momentum fraction of the partons to $x_i \to 1$, $i = 1, 2$.

At the partonic level the Sudakov limit is $\tau \to 1$. It is important to understand that not all the terms enhanced in this limit are logarithmic. For kinematical reasons the multi-loop virtual corrections are all proportional to $\delta(1-\tau)$. The real-emission corrections are either proportional to $1/(1-\tau)$, singular in the infrared limit, or are regular functions of τ, suppressed (not enhanced) at threshold. The sum of the infrared real and infrared virtual radiation defines the $1/(1-\tau)_+$ distributions, source of the large logarithms. The pure hard-virtual corrections generate residual $\delta(1-\tau)$ terms, which are enhanced but do not need to be resummed. We can then decompose the partonic cross section as

$$Q^2 \frac{d\hat{\sigma}^{\text{DY}}_{a_1 a_2}}{dQ^2}(\tau, Q^2, \mu_F^2) = C^{\text{DY}}_{a_1 a_2}(\alpha_{\text{S}}(\mu_R^2); Q^2)\, W_{a_1 a_2}(\alpha_{\text{S}}(\mu_R^2); \tau, Q^2, \mu_F^2), \quad (3.12)$$

where

$$W_{a_1 a_2}(\alpha_{\text{S}}; \tau, Q^2, \mu_F^2) = \delta(1-\tau)\, \delta_{a_1 q}\delta_{a_2 \bar{q}} + \sum_{n=1}^{\infty} \left(\frac{\alpha_{\text{S}}}{\pi}\right)^n W^{(n)}_{a_1 a_2}(\tau, Q^2, \mu_F^2)\,, \quad (3.13)$$

$$C^{\text{DY}}_{a_1 a_2}(\alpha_{\text{S}}; Q^2) = \sigma_0^{\text{DY}}(\alpha_{\text{S}}; Q^2) \left[1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_{\text{S}}}{\pi}\right)^n C^{\text{DY}(n)}_{a_1 a_2}(Q^2)\right]. \quad (3.14)$$

The function W corresponds to the higher-order corrections from soft-gluon radiation. It can be inferred from the universal behaviour of the QCD infrared radiation. The function C^{DY} is the hard-virtual function. It is process-dependent and it is free of logarithmic terms. However, it is not suppressed at threshold: it plays a role already at the NLL accuracy, thanks to the interference with LL terms. In the following we focus on the resummation of the infrared function and we forget about the hard-virtual function. The latter can be calculated order-by-order in perturbation theory, by expanding the resummed formula and comparing it with the fixed-order result.

In the region of interest only incoming quarks give rise to enhanced terms at $\tau \to 1$, because soft fermion emission is not IR singular. Since soft-gluon radiation is flavour conserving, only the flavour non-singlet part of W matters. We can safely set the flavour indices a_1, a_2 to $q, \bar{q}$. In the centre-of-mass frame of p_1 and p_2 where the energy of the initial-state partons is $p_1^0 = p_2^0 = E \sim \sqrt{Q^2}/2$, the first-order radiative correction in the eikonal limit is

$$\begin{aligned} W^{(1)}_{q\bar{q}}(\tau, Q^2, \mu_F^2) = &-\int dw^{(1)}_{\text{DY}}(q)\, \delta(1-\tau) \\ &+ \int dw^{(1)}_{\text{DY}}(q)\, \delta\left(1 - \tau - \frac{\omega}{E}\right) \Theta(q_{\text{T}}^2 - \mu_F^2), \end{aligned} \quad (3.15)$$

where q is the momentum of the radiated (real or virtual) gluon and ω is its energy. The dependence of $dw^{(1)}_{\rm DY}$ on Q^2 is left implicit. The first term in the r.h.s. of Eq. (3.15) is the infrared virtual correction, fixed by unitarity. The second term is the real correction, integrated with the kinematical constraint of the single-emission phase-space. The $dw^{(1)}_{\rm DY}$ function is the differential single-emission probability in the soft limit, defined as

$$dw^{(1)}_{\rm DY}(q) = -\frac{d^3q}{4\pi\omega}|\boldsymbol{J}_{\rm DY}(q)|^2 \,. \tag{3.16}$$

The eikonal factor $\boldsymbol{J}^{\mu}_{\rm DY}$ for the quark-antiquark channel of the Drell-Yan process is

$$\boldsymbol{J}^{\mu}_{\rm DY}(q) = \frac{p_1^{\mu}}{p_1\cdot q}\mathbf{T}_1 + \frac{p_2^{\mu}}{p_2\cdot q}\mathbf{T}_2 \,. \tag{3.17}$$

as defined in Chapter 2 by Eqs. (2.72) and (2.74). From colour conservation it follows that

$$0 = (\mathbf{T}_1+\mathbf{T}_2)^2 = C_{a_1} + C_{a_2} + 2\,\mathbf{T}_1\cdot\mathbf{T}_2 = 2\,(C_F + \mathbf{T}_1\cdot\mathbf{T}_2)\,. \tag{3.18}$$

so that $\mathbf{T}_1\cdot\mathbf{T}_2 = -C_F$, where the scalar C_F is the Casimir operator of the quark. The squared current is then diagonal in colour space,

$$|\boldsymbol{J}_{\rm DY}(q)|^2 = -C_F\,\frac{2p_1\cdot p_2}{p_1\cdot q\, p_2\cdot q}\,. \tag{3.19}$$

The result for higher-order corrections is more involved. Non-abelian gluon correlations are present and they spoil the simple all-order exponentiation characteristic of an abelian theory like QED. As explained in Chapter 2, these correlations cancel by gauge invariance in the leading IR approximation and the exponentiation takes place. The phase-space of this observable factorise in Mellin space. The Mellin transform at fixed $\tau_{\rm H}$ of the hadronic cross-section in Eq. (3.10) is equal to the standard product of the Mellin transforms of the parton densities and the partonic cross-section W. If N is the variable conjugated to $\tau_{\rm H}$, then the N-moment of the cross-section is

$$\begin{aligned} Q^2\frac{d\sigma_N^{\rm DY}}{dQ^2}(Q^2) = \big\{ & f_{q/h_1,N}(\mu_F^2) f_{\bar q/h_2,N}(\mu_F^2) + f_{\bar q/h_1,N}(\mu_F^2) f_{q/h_2,N}(\mu_F^2)\big\} \\ & \times \left[C^{\rm DY}_{q\bar q}\, W_{q\bar q,N}\right](Q^2,\mu_F^2) + \mathcal{O}(1/N)\,. \end{aligned} \tag{3.20}$$

The $\tau_{\rm H}\to 1$ threshold limit corresponds to the $N\to\infty$ limit in Mellin space. In this region, the non-soft real corrections, that are regular in $\tau_{\rm H}$, are suppressed as inverse powers of N. In the r.h.s. of Eq. (3.20) the (inverse) Sudakov parameter N is conjugated to x_1, x_2, τ, which are then forced to 1 in the large-N limit. This is consistent with the kinematical constraints. The hadronic cross section $d\sigma$ and the infrared function W are power series of α_S. The aim is to replace the fixed-order perturbative expansion of $W(\alpha_S)$ with an improved (resummed) function. In the leading IR approximation the LL terms are resummed by simply exponentiating the single-emission probability. Note that the maximum value of the transferred momentum q^2 is $(1-z)Q^2$, which is very different from Q^2 in the Sudakov limit. In terms of the energy fraction $z = 1 - \omega/E$ and of the transverse momentum $q_T^2 \sim (1-z)q^2$, one gets

$$\begin{aligned}
&\ln W_{q\bar{q},N}(Q^2, \mu_F^2) \\
&\quad = \int dw_{\text{DY}}^{(1)}(q) \frac{\alpha_S(q_T^2)}{\pi} \left[\left(1 - \frac{\omega}{E}\right)^{N-1} - 1 \right] \Theta(E - \omega)\, \Theta(q_T^2 - \mu_F^2) \\
&\quad = 2\, \frac{C_F}{\pi} \int_0^1 dz\, \frac{z^{N-1} - 1}{1 - z} \int_{\mu_F^2}^{(1-z)^2 Q^2} \frac{dq_T^2}{q_T^2}\, \alpha_S(q_T^2) + \mathcal{O}\left((\alpha_S \ln N)^n \right). \qquad (3.21)
\end{aligned}$$

We note that the scale of the running coupling is set by the transverse momentum q_T^2 of the soft-gluon and not by its virtuality q^2. It has been shown [70] that this choice resums leading IR singularities. According to the renormalisation group equation, the rescaling $\alpha_S(q^2) \to \alpha_S((1-z)q^2)$ corresponds to

$$\alpha_S((1-z)q^2) = \alpha_S(q^2) \left[1 - \frac{\beta_0}{2\pi}\, \alpha_S(q^2) \ln(1-z) + \mathcal{O}(\alpha_S^2) \right], \qquad (3.22)$$

where $\beta_0 = 11/6\, C_A - 1/3\, n_F$. The terms in the square brackets belong to a power series of $\alpha_S \ln(1-z)$, corresponding to a power series in $\alpha_S \ln(N)$ once the Mellin transform in Eq. (3.21) is worked out. The impact of this correction is to be found at all the logarithmic levels (also the LL).

A comment on the dynamics of the Drell-Yan process is now in order. The singular IR terms in Eq. (3.21), which are responsible for the large logarithms, are due to *soft* radiation from the initial-state partons (quarks). Unresolved radiation include soft-collinear gluons, large-angle soft gluons and hard-collinear gluons. However, if a hard gluon is radiated from the initial-state, then the momentum loss would be non-negligible and $Q^2 \ll s$. Such a contribution exists and has the same LO kinematics of the Born process, but it belongs to the region $\tau \ll 1$ and hence $\tau_H \ll 1$, away from threshold. In Mellin space, it is a power-suppressed term. This is of great importance in order to improve the logarithmic accuracy of the resummed result. The inclusion of the single-logarithmic terms is restricted to the evaluation of the soft-gluon eikonal factor to higher orders. The collinear spectrum is the same as for the single-emission probability. Subleading terms can be formally included with the replacement

$$\frac{\alpha_S}{\pi} C_F \to A_q^{\rm th}(\alpha_S)\,, \tag{3.23}$$

where the function $A_a^{\rm th}$ $(a = q, g)$ has the perturbative expansion

$$A_a^{\rm th}(\alpha_S) = \frac{\alpha_S}{\pi} C_a + \sum_{n=2}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n A_a^{{\rm th}(n)}\,. \tag{3.24}$$

The first term of the series in Eq. (3.24) comes from the soft-collinear emission from parton a. For the Drell-Yan process it must be $a = q$ at threshold, but the following arguments are independent of the flavour of the hard emitter. Higher-order terms in Eq. (3.24) contribute with additional powers of α_S without logarithmic terms and hence belong to subleading corrections. In particular, the $A_a^{{\rm th}(2)}$ coefficient is needed to reach a NLL accuracy. It can be inferred by expanding the resummed formula at order $\mathcal{O}(\alpha_S^2)$ and comparing the result with from a fixed-order NNLO calculation in the IR limit, with the constraint that all the additional radiation is soft with respect to the hard emitter. Under this condition only a soft-collinear gluon, of energy ω_1, is *directly* emitted from the hard parton a, with probability C_a (large-angle soft emission is allowed, but its net contribution is zero in view of colour coherence). A subsequent emission from the soft gluon is described by the full Altarelli-Parisi splitting probability $P_{bg}(1-\omega_2/\omega_1)$, where b is the flavour of the second parton (now it can be a gluon or a quark) and ω_2 is its energy. ω_2 can be soft or hard, with respect to the energy ω_1 of the first soft gluon. The sum over $b = q, g$ and the integral over $\omega_2 = (0, \omega_1)$ lead to the factor $K/2$. In the $\overline{\rm MS}$ scheme,

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R\, n_F\,, \tag{3.25}$$

where n_F is the number of massless quark flavours. The first term in the r.h.s. of Eq. (3.25) is due to a triple gluon vertex and the second one to a $g \to q\bar{q}$ collinear splitting. The $A_a^{{\rm th}(2)}$ coefficient takes into account the soft-collinear emission of a gluon from a together with its subsequent splitting. It is then equal to

$$A_a^{{\rm th}(2)} = \frac{1}{2} C_a K\,. \tag{3.26}$$

The exponentiation of $A_a^{\rm th}(\alpha_S)$ with its $A_a^{{\rm th}(2)}$ coefficient resums all the missing NLL terms in Eq. (3.21). In the iterative picture of parton cascades,

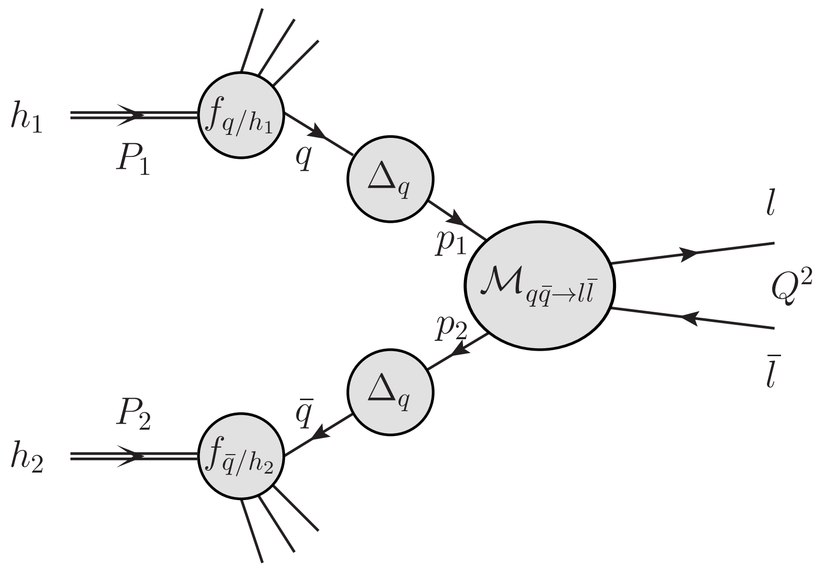

Figure 3.2: Pictorial representation of the all-order resummation formula in Eq. (3.27). All the accompanying radiation is included in the Δ_q radiative factors.

$A_a^{\mathrm{th}(1)}$ accounts for all the double-logarithmic splittings, where each emission in the cascade is soft-collinear, and $A_a^{\mathrm{th}(2)}$ for the residual single-logarithmic collinear branchings. Higher logarithmic accuracies are obtained with the explicit evaluation of higher-order coefficients $A_a^{\mathrm{th}(n)}$. The perturbative coefficients $A_a^{\mathrm{th}(1)}$, $A_a^{\mathrm{th}(2)}$ [35, 37, 43] and $A_a^{\mathrm{th}(3)}$ [71, 72] are explicitly known. The invariance of the hadronic cross section with respect to μ_F variations implies that the function $A_a^{\mathrm{th}}(\alpha_{\mathrm{S}})$ compensates the factorization-scale dependence of the PDFs, as given by the DGLAP evolution equations. It follows that the function $A(\alpha_{\mathrm{S}})$ coincides at any perturbative order with the function that controls the large-N behaviour [73] of the quark ($a = q$) or gluon ($a = g$) anomalous dimension.

At this point we can write a formal *all-order* threshold resummation formula for the Drell-Yan process. We notice that the contribution to Eq. (3.21) of the initial-state quark and the initial-state antiquark must be the same. The overall factor 2 at the exponent accounts for the two collinear regions $q||p_1$ and $q||p_2$. We can write

$$W_{q\bar{q},N}(Q^2, \mu_F^2) = \left[\Delta_{q,N}(Q^2, \mu_F^2)\right]^2 , \tag{3.27}$$

where

$$\Delta_{a,N}(Q^2,\mu_F^2) = \exp\left\{\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\int_{\mu_F^2}^{(1-z)^2Q^2}\frac{dq_{\mathrm{T}}^2}{q_{\mathrm{T}}^2}\,A_a^{\mathrm{th}}\left(\alpha_{\mathrm{S}}(q_{\mathrm{T}}^2)\right)\right\}. \tag{3.28}$$

The radiative factor Δ_a resums the large-logarithmic terms due to soft-collinear emission from parton a. It does so by exponentiating the infrared single-emission probability, calculated at the desired logarithmic accuracy. By using the Renormalisation Group Equation for the running coupling, the integrals in Eq. (3.28) can be explicitly performed at arbitrary logarithmic accuracy (see e.g. Appendix C of [74]) and the radiative factor can be cast in the form of Eq. (3.6), with $L = \ln N$.

Beyond NLL accuracy, the effect of soft-gluon radiation at large angles has to be taken into account and contributes by modifying Eq. (3.27) with the additional factor

$$\Delta_{(\mathrm{int})} = \exp\left\{\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\,D_c(\alpha_{\mathrm{S}}((1-z)^2Q^2)\right\}, \tag{3.29}$$

where the perturbative expansion of the function D_c reads

$$D_c(\alpha_{\mathrm{S}}) = \left(\frac{\alpha_{\mathrm{S}}}{\pi}\right)^2 D_c^{(2)} + \left(\frac{\alpha_{\mathrm{S}}}{\pi}\right)^3 D_c^{(3)} + \mathcal{O}(\alpha_{\mathrm{S}}^4). \tag{3.30}$$

The perturbative coefficients $D_c^{(2)}$ [75, 76] and $D_c^{(3)}$ [77, 78] are explicitly known.

The threshold resummation formula for the Drell-Yan cross section, defined by Eqs. (3.20) and (3.27), assigns a radiative factor to each initial-state parton. We have seen how the kinematics suppresses any hard-collinear emission, in the near-threshold region. This is not always the case. A typical example is that of a fully-inclusive jet production. If the momentum of a final-state parton is not registered, there is no soft constraint on the collinear emission. The final states of the cascade are integrated over anyway and they contribute to the same jet at the hadronic level. The soft-collinear radiative factor of Eq. (3.28) does not contain all the singular terms that are allowed in this limit. Another collinear function is required, in order to resum the extra hard-collinear contribution to all orders in α_{S}.

3.2 The collinear jet function

A simple process with a final-state parton of unconstrained momentum is the deep inelastic scattering. The kinematics is defined at the beginning of Section 2.1. If P is the momentum of the scattering hadron, $p = xP$ the momentum of the initial-state parton and Q that of the space-like photon, then the transferred momentum fraction at the hadronic and partonic level are

$$\tau'_{\rm H} = \frac{Q^2}{2P\cdot Q}\,, \qquad z = \frac{Q^2}{2p\cdot Q} = \frac{\tau'_{\rm H}}{x}\,. \tag{3.31}$$

Note that Q^2 is defined as $Q^2 = -Q_\mu Q^\mu > 0$. In order to compare the DY and DIS processes we make use of the same factorisation scheme, with the same parton densities and the same renormalisation and factorisation scale μ^2. For simplicity we forget about the lepton current and we define a cross section for the hadron-γ^* scattering,

$$\begin{aligned} Q^2\frac{d\sigma^{\rm DIS}}{dQ^2}(\tau'_{\rm H},Q^2) = \sum_a \int_0^1 & dx\, f_{a/h}(x,\mu_F^2) \\ & \times \int_0^1 dz\,\delta(\tau'_{\rm H}-xz)\,Q^2\frac{d\hat\sigma_a^{\rm DIS}}{dQ^2}\left(\alpha_{\rm S}(\mu^2);z,Q^2,\mu^2\right)\,, \end{aligned} \tag{3.32}$$

where

$$Q^2\frac{d\hat\sigma_a^{\rm DIS}}{dQ^2}(z,Q^2,\mu_F^2) = C_a^{\rm DIS}(\alpha_{\rm S}(\mu_R^2);Q^2)\,F_a(\alpha_{\rm S}(\mu_R^2);z,Q^2,\mu_F^2) \tag{3.33}$$

is the cross section for the partonic reaction

$$a(p) + \gamma^*(Q) \to a(\bar p) + X\,. \tag{3.34}$$

In analogy to Eq. (3.12), the $C^{\rm DIS}$ function includes the process-dependent contribution at the hard scale, while F takes into account the process independent soft or collinear QCD radiation from the partons. The Sudakov region is now approached in the limit $\tau'_{\rm H} \to 1$. At the partonic level it must be $x \to 1$ and $z \to 1$. Apart from higher-order virtual corrections, the gluon does not contribute to the scattering against an excited photon. On the contrary a scattering against a Higgs boson would select the gluon channel. We focus on $F_q = F_{\bar q}$. Since the kinematics now let the scattered parton to radiate hard-collinear partons in the Sudakov limit, it is reasonable to separate the pure-soft and the hard-collinear contributions. The function F can be expressed as a convolution of the functions $F^{\rm soft}$ and $J^{\rm coll}$. The Mellin

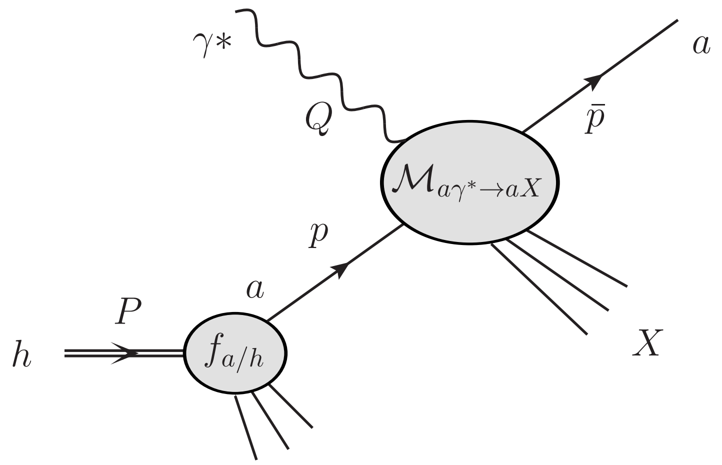

Figure 3.3: Deep inelastic scattering.

transform of the hadronic cross section at fixed momentum transfer $\tau'_{\rm H}$ leads to the factorised expression

$$Q^2 \frac{d\sigma_N^{\rm DIS}}{dQ^2}(Q^2) = \left\{ f_{q/h,N}(\mu_F^2) + f_{\bar{q}/h,N}(\mu_F^2) \right\} \left[C_q^{\rm DIS} \, F_{q,N}^{\rm soft} \, J_{q,N}^{\rm coll} \right](Q^2, \mu_F^2) + \mathcal{O}(1/N) \,. \tag{3.35}$$

The process-dependent terms $\sigma_0^{\rm DIS}$ and $C^{\rm DIS}$ are completely factorised. The soft function $F_N^{\rm soft}$ is analogous, but not equal, to the Drell-Yan function W_N of Eq. (3.20). The collinear function $J_N^{\rm coll}$ is essentially new.

We start with the pure-soft contributions. The calculation of the single-emission probability in the leading IR approximation is similar to that of the previous section. The momentum of the scattered parton is $\bar{p} = p + Q$, light-like only at $z = 1$. The eikonal factor of Eq. (3.19) is replaced by

$$|\boldsymbol{J}_{\rm DIS}(q)|^2 = -C_F \, \frac{2p \cdot \bar{p}}{p \cdot q \, \bar{p} \cdot q} + C_F \, \frac{\bar{p}^2}{(\bar{p} \cdot q)^2} \,. \tag{3.36}$$

The second term in the r.h.s. of Eq. (3.36) is subleading at threshold, since $\bar{p}^2 = Q^2(1-z)/z$. It can be regarded as a non-collinear soft contribution to the time-like evolution of $\bar{p}$, or as a collinear non-soft contribution from the space-like (time-reversed) evolution of the massless final-state parton of momentum $\bar{p}_0 = p + Q - q$. We put this extra contribution into $J^{\rm coll}$. We define $dw_{\rm DIS}^{(1)}$ as in Eq. (3.16), by including only the leading soft-collinear contribution of the eikonal factor, i.e. the first term in the r.h.s. of Eq. (3.36). After the usual expansion in powers of $\alpha_{\rm S}/\pi$, the first-order radiative correction to $F^{\rm soft}$ has the same structure of Eq. (3.15), if E is defined as energy of the initial-state parton in the p, Q rest frame ($2E = \sqrt{Q^2/(1-z)}$). The fixed-order function $F^{\rm soft}(\alpha_{\rm S})$ can be replaced by a resummed expression by following the same steps of the previous sections. The all-order result is

$$F^{\rm soft}_{q,N}(Q^2,\mu_F^2) = \exp\left\{\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\int_{\mu_F^2}^{(1-z)Q^2}\frac{dq_{\rm T}^2}{q_{\rm T}^2}\,A_q^{\rm th}\left(\alpha_{\rm S}(q_{\rm T}^2)\right)\right\}. \quad (3.37)$$

The coefficient of Eq. (3.37) is one half of that in Eq. (3.27). The upper boundary on the transverse-momentum phase-space is also different. In the DIS process the transferred momentum q^2 reaches the Q^2 scale, corresponding to the emission of a hard (collinear) gluon, while it is constrained by $(1-z)Q^2$ in the Drell-Yan process, where only soft-gluon emission is allowed near the threshold region.

The considerations on the factorisation properties of the colour algebra for the Drell-Yan production are valid in this case, too. The large-angle soft radiation vanishes by quantum interference and colour conservation. The full contribution of the initial-state parton to the $F_N^{\rm soft}$ function is still described by the soft-collinear factor Δ_N of Eq. (3.28). The extra logarithms are due to radiation from the scattered parton. We define the $J_N^{\rm soft}$ function such that

$$F^{\rm soft}_{a,N}(Q^2,\mu_F^2) = \Delta_{a,N}(Q^2,\mu_F^2)\,J^{\rm soft}_{a,N}(Q^2)\,, \quad (3.38)$$

where the flavour a can refer to a gluon or a quark. From Eqs. (3.28) and (3.37) it follows that

$$J^{\rm soft}_{a,N}(Q^2) = \exp\left\{-\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\int_{(1-z)Q^2}^{(1-z)^2Q^2}\frac{dq_{\rm T}^2}{q_{\rm T}^2}\,A_a^{\rm th}\left(\alpha_{\rm S}(q_{\rm T}^2)\right)\right\}. \quad (3.39)$$

The exponent of $J_N^{\rm soft}$ has the opposite sign compared to that of Δ_N. In a configuration where initial-state radiation results into Sudakov enhancement, then the final-state jet-like radiation results into Sudakov suppression, and vice versa. For that reason the overall factor 2 of W_N in Eq. (3.21) is not reproduced in F_N, Eq. (3.37). The contribution of the final-state radiator cancels part of the contribution of the initial-state radiator. Besides the sign of the exponent, the behaviour of an undetected parton is very different form that of a registered parton (otherwise a full cancellation would take place and $\ln F_N^{\rm soft}$ would vanish). This account for the mismatch of the non-soft transverse-momentum scale between Eqs. (3.28) and (3.39). The phase-space integral over the parton momentum replaces the convolution with the PDF. The new function is therefore independent of the factorisation scale μ^2. The parton scatters at the energy scale Q^2 of the hard process, corresponding to the transverse-momentum scale $(1-z)Q^2$. Eq. (3.39) describes the scale evolution to the soft-collinear limit $(1-z)^2Q^2$, due to soft emission at vanishing angles. A registered parton in comparison starts to radiate at the hard scale μ^2 at which it 'enters' the scattering process.

We can now examine the emission of non-soft collinear undetected partons, to be comprised in the $J_N^{\rm coll}$ function. It is a NLL function, producing the single-logarithmic terms of the infrared NLO corrections. Without going into the details of the calculation, we consider again the single-emission probability from the Born-level massless emitters p and $\bar{p}_0$. The leading soft-collinear eikonal term must be supplemented by the Altarelli-Parisi splitting probability, in order to match with the non-soft collinear behaviour. The collinear spectrum of each emitter can be isolated via partial fractioning:

$$\frac{p\cdot\bar{p}_0}{p\cdot q\,\bar{p}_0\cdot q}=\frac{p\cdot(p+\bar{p}_0)}{p\cdot q\,(p+\bar{p}_0)\cdot q}+\frac{\bar{p}_0\cdot(p+\bar{p}_0)}{\bar{p}_0\cdot q\,(p+\bar{p}_0)\cdot q}\,. \tag{3.40}$$

The first term in the r.h.s. of Eq. (3.40) is singular in the soft limit and in the region $q||p$. In this collinear region the non-soft terms of the splitting functions are suppressed in the near-threshold limit, because of the kinematical restrictions to initial-state radiation, so that the collinear matching has no effect (it contributes with regular terms). The second term is singular in the soft limit and in the collinear region $q||\bar{p}_0$. Now the non-soft collinear emission contributes with singular subleading terms: by using momentum conservation, i.e. $p+Q=\bar{p}_0+q$, the kinematical constraint yields

$$\delta\left(\bar{p}_0^2\right)=\frac{1}{2p\cdot Q}\,\delta\left((1-z)-\frac{2\bar{p}_0\cdot q}{2p\cdot Q}\right), \tag{3.41}$$

and the threshold limit $z\to 1$ corresponds to $\bar{p}_0\cdot q\to 0$, so that q can be soft but also collinear to $\bar{p}_0$. The bremsstrahlung spectrum for parton a must be replaced with the $P_{ba}(z)$ probability, summed over b and Mellin transformed at fixed z. The momentum to be assigned to the parent parton is $\bar{p}_0/z$, so that the momentum after splitting is $\bar{p}_0$. The leading IR singular term is already included in function $J_N^{\rm soft}$ and must be subtracted from the result. The remaining non-soft collinear term is proportional to $\delta(1-z)$ and its Mellin transform is the flavour-dependent coefficient γ_a, where

$$\gamma_q=\gamma_{\bar{q}}=\frac{3}{2}\,C_F\,,\qquad \gamma_g=\beta_0=\frac{11}{6}\,C_A-\frac{1}{3}\,n_F\,. \tag{3.42}$$

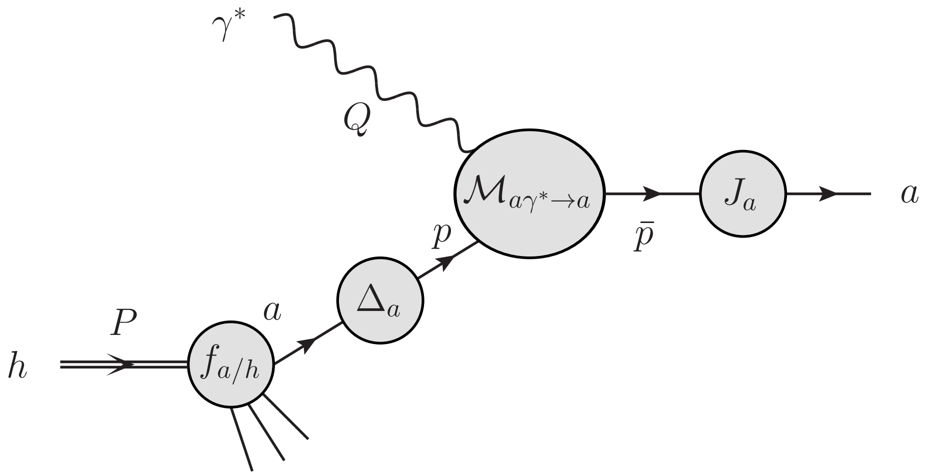

Figure 3.4: Pictorial representation of the all-order resummation formula in Eq. (3.46). All the accompanying radiation is included in the Δ_a radiative factor and in the J_a collinear function.

The exponentiation of the collinear-matched spectrum in the conjugate space up to NNLL terms [35] leads to

$$\ln J^{\rm coll}_{a,N}(Q^2) = -\frac{\gamma_a}{2\pi}\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\,\alpha_{\rm S}((1-z)Q^2)) \\ +\,\mathcal{O}\left(\alpha_{\rm S}(\alpha_{\rm S}\ln N)^n\right). \tag{3.43}$$

The function $J^{\rm coll}$ does not contain any soft interaction. For this reason there is no scale evolution in Eq. (3.43) and the emission cascade occurs at the hard scale $q^2 = Q^2$. As regards the running coupling, the correct scale-choice to include all the NLL terms is the transverse momentum $(1-z)Q^2$ and not the virtuality Q^2. This point has been already discussed in the previous section. Finally, the all-order version of Eq. (3.43) can be obtained with the replacement

$$\frac{\alpha_{\rm S}}{\pi}(-\gamma_a) \to B^{\rm th}_a(\alpha_{\rm S})\,, \tag{3.44}$$

where the function $B^{\rm th}_a$ $(a = q, g)$ has the perturbative expansion

$$B^{\rm th}_a(\alpha_{\rm S}) = \frac{\alpha_{\rm S}}{\pi}(-\gamma_a) + \sum_{n=2}^{\infty}\left(\frac{\alpha_{\rm S}}{\pi}\right)^n B^{{\rm th}(n)}_a\,. \tag{3.45}$$

The lowest term of the expansion accounts for NLL behaviour, the second one for NNLL terms and so on. The soft- and hard-collinear factors for an undetected hard emitter of flavour a can be collected in a single jet function

$J_{a,N}$, such that the IR-singular partonic cross section resummed to all orders is

$$\begin{aligned} F_{a,N}(Q^2,\mu_F^2) &= \Delta_{a,N}(Q^2,\mu_F^2)\, J_{a,N}(Q^2) \\ &= \Delta_{a,N}(Q^2,\mu_F^2)\left[J_{a,N}^{\rm soft}\, J_{a,N}^{\rm coll}\right](Q^2)\,. \end{aligned} \tag{3.46}$$

From Eq. (3.39) and Eqs. (3.43),(3.44) it follows that

$$\begin{aligned} J_{a,N}(Q^2) = \exp\Bigg\{ \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \Bigg[&\int_{(1-z)^2Q^2}^{(1-z)Q^2} \frac{dq_{\rm T}^2}{q_{\rm T}^2}\, A_a^{\rm th}\left(\alpha_{\rm S}(q_{\rm T}^2)\right) \\ &+ \frac{1}{2}\, B_a^{\rm th}\left(\alpha_{\rm S}((1-z)Q^2)\right) \Bigg] \Bigg\}, \end{aligned} \tag{3.47}$$

which essentially describes a small-mass jet recoiling against the Born-level final state.

The soft-collinear factor $\Delta_{a,N}$ and the jet function $J_{a,N}$ are process-independent. They describe the infrared behaviour of any *collinear* emitter a, registered or unregistered, in any QCD process at the partonic threshold. However, the total singular contribution from a radiating parton in semi-inclusive limits includes large-angle soft-gluon radiation and this non-collinear contribution in general does not vanish. It vanishes in scatterings with two hard partons at the LO, like the Drell-Yan and DIS processes. The maximum number of partons for which the colour algebra completely factorises (i.e. the non-collinear contribution cancels out) is three. When four or more hard partons participate to the Born-level scattering, colour-correlations induced by soft-gluon exchanges are described by nontrivial (non-diagonal) colour operators, which can not be reabsorbed in collinear functions. The extension of the resummation formalism to include such contributions is the argument of Chapter 4.

Chapter 4

Single-hadron inclusive production

In this chapter we consider the threshold resummation for a hard-scattering reaction where four hard partons are involved at the partonic level. As discussed in the previous chapter, the soft colour flow between more than three hard partons can not be reduced to the simple scalar factorisation characteristic of collinear emissions. The contribution from large-angle soft-gluon radiation is therefore irreducible and the abstract colour-space formalism is necessary in order to organise the factorisation and exponentiation of all the singular terms.

The specific observable under consideration is the single-hadron inclusive cross section. At sufficiently-large values of the hadron transverse momentum, the cross section for this process factorizes into the convolution of the parton distribution functions of the colliding hadrons with the (short-distance) partonic cross section and with the fragmentation function of the triggered parton into the observed hadron. Since the single inclusive cross section can be easily measured by experiments in hadron collisions, the process offers a relevant test of the QCD factorization picture. Conversely, measurements of the corresponding cross section as function of the transverse momentum and at different collision energies permit to extract quantitative information about the parton fragmentation (especially, the gluon fragmentation) function into the observed hadron, thus complementing the information obtained from hadron production in e^+e^- and lepton-hadron collisions.

The next-to-leading order (NLO) QCD calculation of the cross section for single-hadron inclusive production was completed long ago [79–81]. Soft-gluon resummation of the logarithmically enhanced contributions to the partonic cross section was performed in Ref. [82]. The study of Ref. [82] considers resummation for the transverse-momentum dependence of the cross section integrated over the rapidity of the observed final-state hadron, and it explicitly resums the leading-logarithmic (LL) and next-to-leading logarithmic (NLL) terms. The results of the phenomenological studies (which combine NLL resummation with the complete NLO calculation) in Ref. [82] indicate that the quantitative effect of resummation is rather large, especially in the kinematical configurations that are encountered in experiments at the typical energies of fixed-target collisions.

We first present a general expression for the logarithmically enhanced terms (including the constant term) that correctly reproduces the known NLO result for the transverse-momentum cross section, at fixed rapidity of the observed particle. The calculation is performed by using the soft and collinear approximations of Chapter 2. The result is directly factorised in colour space, and it allows us to explicitly disentangle colour-correlation and colour-interference effects that contribute to soft-gluon resummation at NLL and NNLL accuracy [83].

We then compute the higher-order contributions that dominate near the partonic threshold limit. We employ the formalism of Ref. [44] to write down an all-order resummation formula that controls the logarithmically enhanced contributions. The resummation formula is valid to arbitrary logarithmic accuracy, and it is explicitly worked out up to the NLL level. The general expression of the NLO cross section is required in order to determine the one-loop hard-virtual amplitude that enters into the colour-space factorisation structure of the resummation formula. This is a necessary ingredient to explicitly extend the soft-gluon resummation beyond the next-to-leading logarithmic accuracy.

4.1 NLO results near partonic threshold

In full generality we consider the hard-scattering reaction

$$h_1(P_1) + h_2(P_2) \to h_3(P_3) + X \ , \tag{4.1}$$

where the final states are hadronic. The collision of the hadrons h_1 and h_2 of momenta P_1 and P_2 produces the hadron h_3 of momentum P_3, together with an arbitrary recoiling final state X. The momentum P_3 is completely fixed. Not only the energy and the transverse momentum are known, but also the rapidity of the hadron. According to the QCD factorisation theorem and for large values of the hadron transverse momentum, the cross section for this process factorises into the convolution of the parton distribution functions of the colliding hadrons with the short-distance partonic cross section and with the fragmentation function of the triggered parton into the observed hadron. We use parton densities and fragmentation functions as defined in the $\overline{\text{MS}}$ factorisation scheme, with $f_{a/h}(x,\mu)$ as the parton density of the colliding hadron and $d_{a/h}(x,\mu)$ as the fragmentation function of the parton a into the hadron h. The partonic subprocess corresponding to (4.1) is

$$a_1(p_1) + a_2(p_2) \to a_3(p_3) + X\,, \tag{4.2}$$

where the index $a_i\,(i = 1,2,3)$ denotes the parton species $(a = q, \bar{q}, g)$. The momenta of the initial-state partons are a fraction x_i of the corresponding hadron momenta, $p_i = x_i P_i\,(i = 1,2)$, while the opposite is true for the fragmenting parton, $p_3 = P_3/x_3$. At the Born-level, the initial-state partons produce two final-state particles, one of them with measured transverse momentum. The flavour a_3 is left implicit, in order to describe the contribution of any partonic channel and any final-state system of two particles. For example, if the registered particle is a photon then we expect to recover the results of [43]. If the produced particles are both QCD partons, the triggered one will fragment into the observed hadron h_3, while the other one will generate a recoiling jet (X). At higher orders in perturbation theory and in the near-threshold region, the initial-state partons have just enough energy to produce the triggered final-state parton and a small-mass recoiling jet, formed by collinear partons.

If $d\hat{\sigma}_{a_1a_2\to a_3}(p_1,p_2,p_3)$ is the inclusive cross section for the partonic reaction, the factorisation of the mass singularities leads to

$$\begin{aligned} E_3\frac{d\sigma_{h_3}}{d^3\mathbf{P}_3}(P_1,P_2,P_3) = \sum_{a_1,a_2,a_3}\int_0^1 dx_1dx_2\frac{dx_3}{x_3^2}\, f_{a_1/h_1}^{(x_1,\mu_F)}\, f_{a_2/h_2}^{(x_2,\mu_F)}\, d_{a_3/h_3}^{(x_3,\mu_f)} \\ \times\; p_3^0\frac{d\hat{\sigma}_{a_1a_2\to a_3}}{d^3\mathbf{p}_3}(x_1P_1, x_2P_2, P_3/x_3;\mu_F,\mu_f)\,. \end{aligned} \tag{4.3}$$

The factorisation scale μ_F of the parton densities can be different from the fragmentation scale μ_f. The partonic cross section $d\hat{\sigma}_{a_1a_2\to a_3}$ depends on the factorisation scales. It is computable in QCD perturbation theory as power series expansion in α_S. The perturbative expansion starts at $\mathcal{O}(\alpha_S^2)$

since the leading order partonic process corresponds to the $2 \to 2$ reaction $a_1 a_2 \to a_3 a_4$. Up to the next-to-leading order,

$$d\hat{\sigma}_{a_1 a_2 \to a_3}(p_1, p_2, p_3; \mu_F, \mu_f) = \alpha_S^2(\mu_R^2)\Big(d\hat{\sigma}^{(0)}_{a_1 a_2 \to a_3 a_4}(p_1, p_2, p_3) + \frac{\alpha_S(\mu_R^2)}{2\pi} d\hat{\sigma}^{(1)}_{a_1 a_2 \to a_3}(p_1, p_2, p_3; \mu_R, \mu_F, \mu_f) + \mathcal{O}(\alpha_S^2)\Big) . \quad (4.4)$$

The coefficient of the series are renormalised in the $\overline{\mathrm{MS}}$ scheme, at the renormalisation scale μ_R. The LO term $d\hat{\sigma}^{(0)}_{a_1 a_2 \to a_3 a_4}$ is directly related to the Born-level scattering amplitude of the partonic reaction $a_1 a_2 \to a_3 a_4$. The NLO term $d\hat{\sigma}^{(1)}_{a_1 a_2 \to a_3}$ is known. The contribution of the partonic subprocess with non-identical quarks was computed in Refs. [79, 80], and the complete NLO calculation for all partonic subprocesses was presented in Ref. [81]. The kinematics of the partonic subprocess is usually described in terms of the independent kinematical variables s, v and w, which are related to the customary Mandelstam variables s, t, u as follows

$$v \equiv 1 + t/s \ , \qquad w \equiv -u/(s+t) \, , \quad (4.5)$$

with the corresponding phase-space boundaries

$$s \geq 0 \ , \quad 1 \geq v \geq 0 \ , \quad 1 \geq w \geq 0 \ . \quad (4.6)$$

With massless kinematics, the partonic Mandelstam variables are

$$s = 2p_1 \cdot p_2 \ , \quad t = -2p_1 \cdot p_3 \ , \quad u = -p_2 \cdot p_3 \ . \quad (4.7)$$

Analogous kinematical variables can be introduced for the corresponding hadronic process in Eq. (4.1). For instance, $S = 2P_1 \cdot P_2$ is the square of the centre-of-mass energy of the hadronic collision. Using the s, v, w variables, the partonic cross section in Eqs. (4.3) and (4.4) can be written as

$$p_3^0 \frac{d\hat{\sigma}}{d^3\mathbf{p}_3}(p_1, p_2, p_3; \mu_F, \mu_f) = \frac{\alpha_S^2(\mu_R^2)}{\pi\, s}\Big[\frac{1}{v}\frac{d\hat{\sigma}^{(0)}(s,v)}{dv}\, \delta(1-w) + \frac{\alpha_S(\mu_R^2)}{2\pi}\, \frac{1}{v\, s}\, \mathcal{C}^{(1)}(s, v, w; \mu_R, \mu_F, \mu_f) + \mathcal{O}(\alpha_S^2)\Big] , \quad (4.8)$$

where the flavour indices are left understood. The term in the square bracket exactly corresponds to the square-bracket term in Eq. (10) of Ref. [81], modulo the overall factor $\alpha_S^2(\mu_R^2)$. The first term in the square bracket of Eq. (4.8) is the Born-level contribution and the function $\mathcal{C}^{(1)}$ encodes the NLO corrections.

The Born-level term in Eq. (4.8) has a sharp integrable singularity at $w = 1$. This singularity has a kinematical origin. Indeed $(1 - w)$ is proportional to $s_X = s + t + u$, the invariant mass squared of the unobserved final-state system X in Eqs. (4.1) and (4.2). X comprises the total QCD radiation recoiling against the observed hadron h_3. At the LO, the system X is formed by a single *massless* parton $a_4(p_4)$ and, therefore, $s_X = p_4^2$ exactly vanishes. This leads to the factor $\delta(1 - w)$ in Eq. (4.8). At higher perturbative orders, the LO singularity at $w \to 1$ is enhanced by logarithmic terms of the type $\ln(1 - w)$. The enhancement has a dynamical origin, and it is produced by soft-gluon radiation. In this kinematical region the partonic process in Eq. (4.2) approaches the near-elastic limit, or 'partonic threshold'. The parton a_3 is produced with the maximal energy that is kinematically allowed by momentum conservation, and the recoiling partonic system X is forced to carry a very small invariant mass. As a consequence, the associated production of hard QCD radiation is strongly suppressed. The associated production of soft QCD radiation is instead allowed and, due to the soft-gluon bremsstrahlung spectrum, it generates large logarithmic corrections.

The presence of logarithmically enhanced terms is evident from the known NLO result. The structure of the NLO term $\mathcal{C}^{(1)}$ in Eq. (4.8) is customarily written in the following form:

$$\begin{aligned}\mathcal{C}^{(1)}(s, v, w; \mu_R, \mu_F, \mu_f) = \mathcal{C}_3(v) \left(\frac{\ln(1 - w)}{1 - w}\right)_+ + \mathcal{C}_2(v; s, \mu_F, \mu_f) \left(\frac{1}{1 - w}\right)_+ \\ + \mathcal{C}_1(v; s, \mu_R, \mu_F, \mu_f)\, \delta(1 - w) + \mathcal{O}\left((1 - w)^0\right) . \quad (4.9)\end{aligned}$$

The last term on the right-hand side is a non-singular function of w in the limit $w \to 1$. Explicit expressions in analytic form can be found in Refs. [79, 80]. The functions $\mathcal{C}_3$, $\mathcal{C}_2$ and $\mathcal{C}_1$ do not depend on w, and they multiply functions of w that are singular (and logarithmically enhanced) at $w \to 1$. These singular functions are expressed by $\delta(1 - w)$ and customary plus-distributions $[(\ln^k(1 - w))/(1 - w)]_+$, defined over the range $1 \geq w \geq 0$.

We are interested in the near-threshold behaviour of the NLO corrections, hence on the functions $\mathcal{C}_3, \mathcal{C}_2$ and $\mathcal{C}_1$ in Eq. (4.9). Each of these functions depends on the various flavour channels that contribute to the partonic reaction $a_1a_2 \to a_3a_4$. They are all reported in Sect. 3 of Ref. [81]. The corresponding analytic expressions have a rather involved dependence on v, colour factors and the flavour channel. In Ref. [83] we have presented the results of an independent NLO calculation in the kinematical region close to the partonic threshold. Our results are obtained and expressed in a form that is suitable (and necessary) for the all-order treatment and resummation of the logarithmically enhanced QCD corrections.

To present the our in its factorised form, we employ the representation of the four-parton scattering amplitude in the colour-space notation [66, 67], presented in Section 2.5. The all-loop QCD amplitude $\mathcal{M}$ of the scattering process in Eq. (4.13) is written as

$$|\mathcal{M}_{a_1a_2a_3a_4}\rangle = \alpha_{\rm S}(\mu_R^2)\left[|\mathcal{M}^{(0)}_{a_1a_2a_3a_4}\rangle + \sum_{n=1}^{\infty}\left(\frac{\alpha_{\rm S}(\mu_R^2)}{2\pi}\right)^n |\mathcal{M}^{(n)}_{a_1a_2a_3a_4}(\mu_R)\rangle\right], \tag{4.10}$$

where $\mathcal{M}^{(0)}$ is the Born-level contribution, $\mathcal{M}^{(n)}$ is the renormalised contribution at the n-loop level. The dependence on the parton momenta p_i ($i = 1, \ldots, 4$) is not explicitly denoted. Real and virtual gluon radiation from the parton with momentum p_i is described by the colour-charge matrix $(\mathbf{T}_i)^c$ (c is the colour index of the radiated gluon) and colour conservation implies

$$\sum_{i=1}^{4} \mathbf{T}_i\, |\mathcal{M}_{a_1a_2a_3a_4}\rangle = 0\,. \tag{4.11}$$

According to this notation the colour flow is treated as 'outgoing', so that $\mathbf{T}_3$ and $\mathbf{T}_4$ are the colour charges of the partons a_3 and a_4, while $\mathbf{T}_1$ and $\mathbf{T}_2$ are the colour charges of the anti-partons $\overline{a}_1$ and $\overline{a}_2$. Non-trivial colour correlations are produced by the quadratic operators $\mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i$ with $i \neq j$. With $i, j = 1, \ldots, 4$ there are six different operators, but, due to colour conservation (i.e. Eq. (4.11)), only two of them lead to colour correlations that are linearly independent (see the Appendix A of Ref. [66]). Two linearly independent operators are e.g. $\mathbf{T}_1 \cdot \mathbf{T}_3$ and $\mathbf{T}_2 \cdot \mathbf{T}_3$. Different choices of pairs of independent operators are feasible and physically equivalent.

The elastic $2 \to 2$ process is evaluated exactly at the partonic threshold ($s + t + u = 0$), and momentum conservation ($p_1 + p_2 = p_3 + p_4$) implies that $\mathcal{M}$ only depends on two kinematical variables, e.g. s and v. The LO cross section in Eq. (4.8)) depends on the square of the Born-level scattering amplitude $|\mathcal{M}^{(0)}\rangle$,

$$\frac{d\hat{\sigma}^{(0)}_{a_1a_2\to a_3a_4}(s,v)}{dv} = \frac{1}{N^{(in)}_{a_1a_2}}\,\frac{1}{16\pi\, s}\,|\mathcal{M}^{(0)}_{a_1a_2a_3a_4}|^2\,, \tag{4.12}$$

where $|\mathcal{M}^{(0)}|^2 = \langle\mathcal{M}^{(0)} \mid \mathcal{M}^{(0)}\rangle$ and the factor $N^{(in)}_{a_1a_2} = 4\, n_c(a_1)\, n_c(a_2)$ comes from the average over the spins and colours ($n_c(q) = n_c(\bar{q}) = N_c, n_c(g) = N_c^2 - 1$) of the initial-state partons a_1 and a_2.

At the NLO, the parton cross section receives contributions from two types of partonic processes. The elastic process

$$a_1(p_1) + a_2(p_2) \to a_3(p_3) + a_4(p_4)\,, \tag{4.13}$$

which has to be evaluated with one-loop virtual corrections, and the inelastic process in Eq. (4.2) with real emission of $X = \{2 \text{ partons}\}$, which is evaluated at the tree level. Virtual and real contributions are separately divergent, and we use conventional dimensional regularisation (CDR) [84] in $d = 4-2\epsilon$ space-time dimensions to deal with both ultraviolet and infrared divergences.

The elastic process contributes only to the term proportional to $\delta(1-w)$ in Eq. (4.9), and its contribution is directly proportional to the renormalised one-loop scattering amplitude of the four-parton process. The one-loop scattering amplitude includes IR-divergent terms that have a process-independent (universal) structure [67, 85]. In order to present the result of our calculation we need to provide a definition of the finite part of the virtual amplitude. The NLO contribution $\mathcal{C}^{(1)}$ in Eq. (4.8) depends on the IR-finite part $\mathcal{M}^{(1)\,\mathrm{fin}}$ of the one-loop scattering amplitude. The IR-finite part is obtained through the factorisation formula

$$|\mathcal{M}^{(1)}\rangle = \mathbf{I}^{(1)}_{\mathrm{sing}}\, |\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(1)\,\mathrm{fin}}\rangle\,, \tag{4.14}$$

where the colour operator $\mathbf{I}^{(1)}_{\mathrm{sing}}$ embodies the one-loop IR divergence in the form of double and single poles ($1/\epsilon^2$ and $1/\epsilon$), while $\mathcal{M}^{(1)\,\mathrm{fin}}$ is finite as $\epsilon \to 0$. The Born-level and one-loop ($|\mathcal{M}^{(1)}\rangle$) scattering amplitudes of the partonic reaction $a_1 a_2 \to a_3 a_4$ are known [86, 87]. To specify the expression of $\mathcal{M}^{(1)\,\mathrm{fin}}$ in an unambiguous way, the contributions of $\mathcal{O}(\epsilon^0)$ that are included in $\mathbf{I}^{(1)}_{\mathrm{sing}}$ must be explicitly defined. We use the expression

$$\begin{aligned}\mathbf{I}^{(1)}_{\mathrm{sing}} = \frac{1}{2}\frac{1}{\Gamma(1-\epsilon)}\Bigg[&\frac{1}{\epsilon^2}\sum_{\substack{i,j=1\\ i\neq j}}^{4} \mathbf{T}_i\cdot\mathbf{T}_j \left(\frac{4\pi\mu_R^2\, e^{-i\lambda_{ij}\pi}}{2p_i\cdot p_j}\right)^{\epsilon} \\ &- \frac{1}{\epsilon}\sum_{i=1}^{4}\gamma_{a_i}\left(\frac{4\pi\mu_R^2\, s}{u\, t}\right)^{\epsilon}\Bigg],\end{aligned} \tag{4.15}$$

where $e^{-i\lambda_{ij}\pi}$ is the unitarity phase factor ($\lambda_{ij} = -1$ if i and j are both incoming or outgoing partons and $\lambda_{ij} = 0$ otherwise). The flavour dependent coefficients γ_a are

$$\gamma_q = \gamma_{\bar q} = \frac{3}{2}C_F \quad , \qquad \gamma_g = \frac{11}{6}C_A - \frac{1}{3}n_F\,. \tag{4.16}$$

As regards the real corrections in the threshold region $w \to 1$, the inelastic process of Eq. (4.2) gives dominant NLO contributions only from two kinematical configurations of the system $X = \{2 \text{ partons}\}$: either one of the two partons is soft or both partons are collinear. We have treated these two configurations by using the soft and collinear factorisation formulae of Section 2.5 for the scattering amplitudes. The real emission term is finally combined with the collinear-divergent counterterms necessary to define the NLO parton densities and fragmentation function.

The final result of our NLO calculation is IR finite and has a factorised structure. It is given in terms of flavour and colour-space factors that acts on the scattering amplitude of the four-parton elastic process. Up to regular $\mathcal{O}\left((1-w)^0\right)$ terms,

$$16\pi\, N^{(in)}\, \mathcal{C}^{(1)} = \langle \mathcal{M}^{(0)} |\, \boldsymbol{C}^{(1)} | \mathcal{M}^{(0)} \rangle + \left(\langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)\,\mathrm{fin}} \rangle + \mathrm{c.c.} \right) \delta(1-w)\,, \tag{4.17}$$

where c.c. stands for complex conjugate, and the flavour indices are left understood. The function $\boldsymbol{C}^{(1)}$ is the colour-space operator

$$\begin{aligned}
\boldsymbol{C}^{(1)}_{a_1 a_2 a_3 a_4}(s,v,w;\mu_R,\mu_F,\mu_f) &= 2\left(\frac{\ln(1-w)}{1-w}\right)_+ \left[2\sum_{i=1}^{3} \mathbf{T}_i^2 - \mathbf{T}_4^2 \right] \\
&- \left(\frac{1}{1-w}\right)_+ \left[2\sum_{i=1}^{3} \mathbf{T}_i^2 \left(\ln\frac{1-v}{v} + \ln\frac{\mu_{Fi}^2}{s} \right) - 2\,\mathbf{T}_4^2 \ln(1-v) \right. \\
&\left. + \gamma_{a_4} + 8\left(\mathbf{T}_1\cdot\mathbf{T}_3 \ln(1-v) + \mathbf{T}_2\cdot\mathbf{T}_3\, \ln v \right) \right] \\
&+ \delta(1-w) \left\{ \frac{\pi^2}{2} \left(\mathbf{T}_1^2 + \mathbf{T}_2^2 + 3\mathbf{T}_3^2 - \frac{4}{3}\mathbf{T}_4^2 \right) \right. \\
&- 2\mathbf{T}_3^2 \ln v\, \ln\frac{\mu_f^2}{s} + 2\mathbf{T}_2^2 \ln\frac{1-v}{v} \ln\frac{\mu_F^2}{s} - \sum_{i=1}^{3} \gamma_{a_i} \ln\frac{\mu_{Fi}^2}{s\,v(1-v)} \\
&+ \ln v \ln(1-v) \left(\mathbf{T}_4^2 - \mathbf{T}_1^2 - \mathbf{T}_2^2 - \mathbf{T}_3^2 \right) \\
&+ \ln^2(1-v) \left(\mathbf{T}_1^2 + \mathbf{T}_3^2 - \mathbf{T}_4^2 \right) + \ln^2 v \left(\mathbf{T}_2^2 + \mathbf{T}_3^2 \right) + \gamma_{a_4} \ln(1-v) \\
&+ \mathbf{T}_1\cdot\mathbf{T}_3 \left(2\pi^2 + 2\ln(1-v)\left(\ln(1-v) - 2\ln v\right) \right) \\
&\left. + \mathbf{T}_2\cdot\mathbf{T}_3 \left(2\pi^2 + 2\ln v\left(2\ln(1-v) - 3\ln v\right) \right) + K_{a_4} \right\}.
\end{aligned} \tag{4.18}$$

In Eq. (4.17) we have introduced the factorisation scales

$$\mu_{F1} = \mu_{F2} = \mu_F \quad , \qquad \mu_{F3} = \mu_f\,, \tag{4.19}$$

and the flavour dependent coefficients

$$K_q = K_{\bar{q}} = \left(\frac{7}{2} - \frac{\pi^2}{6} \right) C_F \quad , \qquad K_g = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} n_F \,. \tag{4.20}$$

The colour operator in Eq. (4.18) contains terms that are proportional to plus-distributions of w and a term that is proportional to $\delta(1-w)$. The action of the former terms onto the Born-level scattering amplitude as in Eq. (4.17) directly gives the coefficients $\mathcal{C}_3$ and $\mathcal{C}_2$ in Eq. (4.9). The sum of the latter term and the analogous term (which is proportional to $\mathcal{M}^{(1)\,\mathrm{fin}}$) on the right-hand side of Eq. (4.17) gives the function $\mathcal{C}_1$ in Eq. (4.9). Note that a change in the definition of $\mathcal{M}^{(1)\,\mathrm{fin}}$ would be compensated by a corresponding change in $\boldsymbol{C}^{(1)}$, so that the total NLO result in Eqs. (4.9) and (4.17) is unchanged.

All the contributions to the NLO colour-space function $\boldsymbol{C}^{(1)}$ in Eq. (4.18) have a definite physical origin. The terms that are proportional to the colour charges $\mathbf{T}_i$ are due to radiation (either collinear or at wide angles) of soft gluons. In particular, the coefficients of $(1/(1-w))_+$ and $\delta(1-w)$ depend on colour correlation operators. In Eq. (4.18), we have used the two linearly independent operators $\mathbf{T}_1 \cdot \mathbf{T}_3$ and $\mathbf{T}_2 \cdot \mathbf{T}_3$ to explicitly present the colour correlation contributions. The terms that are proportional to the flavour-dependent coefficients γ_a and K_a have a non-soft collinear origin. In particular, we recall (see Eq. (C.13) in Appendix C of Ref. [66]) that K_a is related to the $(d-4)$-dimensional part (i.e. the terms of $\mathcal{O}(\epsilon)$) of the LO collinear splitting functions. We also remark that the gluonic coefficient K_g in Eq. (4.20) is exactly equal to the coefficient K in Eq. (3.25) that controls the intensity of soft-gluon radiation at $\mathcal{O}(\alpha_S^2)$.

4.2 All-order soft-gluon resummation

We remark that our factorised expression, by keeping explicitly under control colour-correlation effects, embodies an amount of process-independent information that cannot be easily extracted from the results of Ref. [81]. The knowledge of the abstract colour structure is essential to compute all the logarithmically enhanced contributions beyond the NLO. Indeed, the colour interference between this one-loop amplitude and the NLL terms explicitly determines an entire class of resummed contributions at NNLL accuracy.

To discuss the all-order resummation, we introduce the three independent kinematical variables $\{ x_\omega, r, p_T^2 \}$ that are defined by

$$x_\omega = -\frac{u+t}{s} \ , \qquad r = \frac{u}{t} \ , \qquad p_T^2 = \frac{ut}{s} \ , \tag{4.21}$$

with the corresponding phase-space boundaries

$$1 \geq x_\omega \geq 0 \ , \qquad r \geq 0 \ , \qquad p_T^2 \geq 0 \, . \tag{4.22}$$

The variable p_T is the transverse momentum of the observed parton a_3. In the centre-of-mass frame of the partonic collision in Eq. (4.2), the variable $x_\omega = 2\, p_3^0/\sqrt{s}$ is the energy fraction of the parton a_3 and $r = (1+\cos\theta_{13}^*)/(1-\cos\theta_{13}^*)$ is related to its scattering angle θ_{13}^*. The relation with the transverse momentum and rapidity of the parton a_3 is

$$x_\omega = \frac{2\, p_T}{\sqrt{s}} \cosh\eta \ , \qquad r = e^{2\eta} \, . \tag{4.23}$$

The three independent kinematical variables $\{ s, v, w \}$ are not particularly suitable for an all-order treatment near threshold, because of their degree of asymmetry under the exchange $u \leftrightarrow t$. The all-order treatment of the terms $\ln^n(1-w)$ would unavoidably produce an asymmetry with respect to $u \leftrightarrow t$ (see Eq. (4.5)). Any feasible resummed calculations involve the truncation of the all-order series to some level of logarithmic accuracy and, in this case, the asymmetry effect is suppressed only by subleading logarithmic contributions. In practical applications of resummation, this feature can lead to non-negligible and unphysical asymmetries in the angular distribution of the produced hadron h_3.

In terms of the kinematical variables in Eq. (4.21), the near-threshold limit corresponds to the region where $x_\omega \to 1$, at fixed values of p_T and r. Therefore, the threshold variable is x_ω, symmetric with respect to the exchange $u \leftrightarrow t$. The change of variables $\{ s, v, w \} \leftrightarrow \{ x_\omega, r, p_T^2 \}$ can be straightforwardly applied to any smooth functions of these variables. Note that singular plus-distributions require a slightly more careful treatment, because of the presence of contact terms at the endpoints $w = 1$ and $x_\omega = 1$. We have

$$\begin{aligned}
\delta(1-x_\omega) &= \frac{1}{v}\,\delta(1-w) \ , \\
\left(\frac{1}{1-x_\omega}\right)_+ &= \frac{1}{v}\left\{ \left(\frac{1}{1-w}\right)_+ + \delta(1-w)\ln v \right\} , \\
\left(\frac{\ln(1-x_\omega)}{1-x_\omega}\right)_+ &= \frac{1}{v}\left\{ \left(\frac{\ln(1-w)}{1-w}\right)_+ + \left(\frac{1}{1-w}\right)_+ \ln v + \frac{1}{2}\,\delta(1-w)\,\ln^2 v \right\} .
\end{aligned} \tag{4.24}$$

Using Eq. (4.24), the change of variables of Eq. (4.21) can be applied to the complete NLO cross section in Eq. (4.8) and to the NLO results in Eqs. (4.17) and (4.18). We write the all-order partonic cross section in Eqs. (4.3) and (4.4) in the following form:

$$p_3^0 \frac{d\hat{\sigma}_{a_1 a_2 \to a_3}}{d^3\mathbf{p}_3} = \frac{1}{s}\, \sigma^{(0)}_{a_1 a_2 \to a_3 a_4}(r, p_T^2)\; \Sigma_{a_1 a_2 \to a_3}(x_\omega, r; p_T^2, \mu_F, \mu_f) \;\; , \tag{4.25}$$

where the Born-level cross section $\sigma^{(0)}$ is

$$\sigma^{(0)}_{a_1 a_2 \to a_3 a_4}(r, p_T^2) \equiv \frac{\overline{|\mathcal{M}^{(0)}_{a_1 a_2 a_3 a_4}|^2}}{16\pi^2 s} \, , \tag{4.26}$$

and $\overline{|\mathcal{M}^{(0)}|^2}$ denotes the average of $|\mathcal{M}^{(0)}|^2$ over the spins and colours of the initial-state partons a_1 and a_2. The QCD radiative corrections are embodied in the function $\Sigma_{a_1 a_2 \to a_3}$,

$$\begin{aligned} \Sigma_{a_1 a_2 \to a_3}(x_\omega, r; p_T^2, \mu_F, \mu_f) &= \alpha_S^2(\mu_R^2) \Bigg[\delta(1 - x_\omega) \\ &+ \sum_{n=1}^{+\infty} \left(\frac{\alpha_S(\mu_R^2)}{2\pi} \right)^n \Sigma^{(n)}_{a_1 a_2 \to a_3}(x_\omega, r; p_T^2, \mu_R, \mu_F, \mu_f) \Bigg] . \end{aligned} \tag{4.27}$$

Note that the LO factor $\alpha_S^2(\mu_R^2)$ is included in the overall normalisation of Σ and, therefore, the radiative function Σ is renormalisation group invariant. The explicit dependence on μ_R appears only by expanding Σ in powers of $\alpha_S(\mu_R^2)$, as in Eq. (4.27). We also introduce the definition of the Mellin space N-moments Σ_N of the function $\Sigma(x_\omega)$, with respect to the variable x_ω, at fixed values of r and p_T^2,

$$\Sigma_{a_1 a_2 \to a_3, N}(r; p_T^2, \mu_F, \mu_f) \equiv \int_0^1 dx_\omega\; x_\omega^{N-1}\; \Sigma_{a_1 a_2 \to a_3}(x_\omega, r; p_T^2, \mu_F, \mu_f) \, . \tag{4.28}$$

The hard scale of the partonic process is related to p_T^2 rather than to s. The N moment of the singular plus-distribution $\left[\ln^k(1-x_\omega)/(1-x_\omega)\right]_+$ gives $\ln^{k+1} N$ plus additional subleading logarithms of N. The resummation of terms with singular distributions of x_ω corresponds to the resummation of terms with powers of $\ln N$ in Mellin space.

Neglecting contributions of $\mathcal{O}(1/N)$ that are subdominant in the near-threshold limit, we write the radiative function in Eq. (4.28) in the following form:

$$\Sigma_{a_1 a_2 \to a_3, N}(r; p_T^2, \mu_F, \mu_f) = \Sigma^{\rm res}_{a_1 a_2 \to a_3 a_4, N}(r; p_T^2, \mu_F, \mu_f) + \mathcal{O}(1/N) \;\; , \tag{4.29}$$

where $\Sigma_N^{\rm res}$ includes the *all-order* resummation of the $\ln N$ terms. Some corrections of $\mathcal{O}(1/N)$ can also be included in $\Sigma_N^{\rm res}$. In our resummation treatment, the factorisation scales μ_F and μ_f do not play any specific role. The dependence on the factorisation scales and on the renormalisation scale μ_R is treated as in customary perturbative calculations at fixed order and the values of μ_F, μ_f and μ_R have to be set to some scale of the order of $P_T = P_{3T}$, the transverse momentum of the observed hadron. The function $\Sigma_N^{\rm res}$ is analogous to the radiative functions W_N and F_N defined in Eqs. (3.27) and (3.46), for the Drell-Yan and DIS processes respectively. It plays the same role, by resumming all the large logarithmic terms. The structure of $\Sigma_N^{\rm res}$ is obviously richer (starting at the NLL), because of the four-parton exchange of soft gluons emitted at large angles. In addition, the process-dependent terms from the virtual corrections can not be collected anymore into a simple scalar factor, as it was done with $C^{\rm DY}$ and $C^{\rm DIS}$ in Eqs. (3.12) and (3.33). The interference between the multiloop amplitude and the new NLL terms generates NNLL terms and is therefore part of the resummed expression, factorised in colour space.

The all-order expression of $\Sigma_N^{\rm res}$ is obtained by using the techniques of Ref. [44], which treat soft-gluon resummation in quite general terms. The BCMN resummation formulae [44] apply to arbitrary multiparton hard-scattering processes and to general observables that are sensitive to soft-gluon radiation. The dependence on the specific observable is parametrized by a Sudakov weight $u(q)$, which is a purely kinematical function. As discussed in the final part of Ref. [44], in our case of single-particle inclusive production near threshold, the Sudakov weight is simply $u(q) = \exp\{-N(q\cdot p_4)/(p_1\cdot p_2)\}$, where p_4 is the momentum of the recoiling parton a_4 in the elastic-scattering subprocess of Eq. (4.13). Using this expression for $u(q)$ in the BCMN resummed formulae, we directly obtain the resummed expression

$$\Sigma^{\rm res}_{a_1a_2\to a_3a_4,N}(r;p_T^2,\mu_F,\mu_f) = \\ \left[\prod_{i=1,2,3} \Delta_{a_i,N_i}(Q_i^2;\mu_{Fi}^2)\right] J_{a_4,N_4}(Q_4^2)\,\frac{\langle \mathcal{M}_H | \mathbf{\Delta}_N^{(\rm int)}(r;p_T^2)|\mathcal{M}_H\rangle}{|\mathcal{M}^{(0)}|^2}\,, \quad (4.30)$$

where $\mathcal{M}_H$ depends on the flavour indices a_i ($i = 1,\ldots,4$), on the kinematical variables r and p_T^2, and on the factorisation scales μ_F and μ_f. Each factor in the right-hand side of Eq. (4.30) is separately renormalisation group invariant.

The three radiative factors $\Delta_{a_i,N}$ ($i = 1,2,3$) in the right-hand side of Eq. (4.30) embody soft-gluon radiation from the triggered partons a_1, a_2 and

a_3 of the partonic process in Eq. (4.2). The N-moment factor $\Delta_{a,N}$ depends on the flavour of the radiating parton a, on the partonic hard scale Q^2, and on the factorisation scale of the corresponding parton density or fragmentation function in the hadronic cross section. It is the same soft-collinear factor that contributes to the Drell-Yan and DIS resummation formula of Section 3. We employ the all-order form of Eq. (3.28). The kernel of the exponent is the perturbative function $A_a^{\rm th}(\alpha_S)$, whose lower-order coefficients $A_a^{\rm th(1)}$ and $A_a^{\rm th(2)}$, defined in Eqs. (3.24) and (3.26), respectively resum the LL and NLL terms due to soft-collinear radiation from parton a.

The jet function J_{a_4,N_4} in Eq. (4.30) includes soft and (flavour conserving) collinear radiation from the parton a_4 that recoils against the observed parton a_3 in the tree-level (or, more generally, elastic scattering) process $a_1a_2 \to a_3a_4$. The jet function $J_{a,N}$, which depends on the flavour of the radiating parton a and on the partonic hard scale Q^2, is defined in Eq. (3.47). The kernel $A_a^{\rm th}(\alpha_S)$ is the same perturbative function as in Eqs. (3.28). The NLL kernel $B_a^{\rm th}(\alpha_S)$ has the same expression as in Eq. (3.45)

The values of N_i and Q_i^2 $(i = 1, \ldots, 4)$ in the argument of the radiative factors Δ and J in Eq. (4.30) depend on r, p_T^2 and on the moment index N of $\Sigma_N^{\rm res}$. The specification of this dependence involves some degree of arbitrariness (see Ref. [44]) that is compensated by a corresponding dependence in the terms $\mathbf{\Delta}_N^{\rm (int)}$ and $\mathcal{M}_H$. We use the Mellin moment values

$$N_1 = N\,\frac{r}{1+r}\,, \quad N_2 = N\,\frac{1}{1+r}\,, \quad N_3 = N\,, \quad N_4 = N\,\frac{r}{(1+r)^2}\,, \qquad (4.31)$$

and the common hard scale

$$Q_i^2 = p_T^2, \qquad i = 1, 2, 3, 4\,, \qquad (4.32)$$

which unambiguously specify the expressions of $\mathbf{\Delta}_N^{\rm (int)}$ and $\mathcal{M}_H$ that are presented below. A different choice corresponds to a different definition of the hard scale and to a different separation between the collinear and the large-angle emission regions.

The colour-space radiative factor $\mathbf{\Delta}_N^{\rm (int)}$ embodies quantum-interference effects that are produced by soft-gluon radiation at large angles with respect to the direction of the momenta p_i $(i = 1, \ldots, 4)$ of the partons in the $2 \to 2$ hard scattering. Its explicit expression is

$$\mathbf{\Delta}_N^{\rm (int)}(r; p_T^2) = \mathbf{V}_N^\dagger(r; p_T^2)\, \mathbf{V}_N(r; p_T^2)\,, \qquad (4.33)$$

where

$$\mathbf{V}_N(r;p_T^2) = P_z \exp\left\{ \int_0^1 dz \, \frac{z^{N-1}-1}{1-z} \, \mathbf{\Gamma}\left(\alpha_{\rm S}\left((1-z)^2 p_T^2\right); r\right) \right\} . \quad (4.34)$$

The soft-gluon anomalous dimension $\mathbf{\Gamma}(\alpha_{\rm S}; r)$ is a colour-space matrix, and the operator P_z denotes z-ordering in the expansion of the exponential matrix. $\bar{\mathbf{V}}_N$ in Eq. (4.33) is obtained from Eq. (4.34) by replacing P_z with $\bar{P}_z$, the operator acting in the opposite order. The anomalous-dimension matrix $\mathbf{\Gamma}(\alpha_{\rm S}; r)$ has the perturbative expansion

$$\mathbf{\Gamma}(\alpha_{\rm S}; r) = \frac{\alpha_{\rm S}}{\pi} \, \mathbf{\Gamma}^{(1)}(r) + \sum_{n=2}^{\infty} \left(\frac{\alpha_{\rm S}}{\pi}\right)^n \mathbf{\Gamma}^{(n)}(r) \ , \quad (4.35)$$

and the explicit expression of the first-order term is

$$\mathbf{\Gamma}^{(1)}(r) = \mathbf{T}_t^2 \, \ln(1+r) + \mathbf{T}_u^2 \, \ln\frac{1+r}{r} + i\pi \, \mathbf{T}_s^2 \quad (4.36)$$

$$= \mathbf{T}_t^2 \left(\ln(1+r) - i\pi\right) + \mathbf{T}_u^2 \left(\ln\frac{1+r}{r} - i\pi\right) + i\pi \sum_{i=1}^{4} C_{a_i} \, . \quad (4.37)$$

Note that $\mathbf{\Gamma}^{(1)}$ includes colour correlations, which we have explicitly expressed in terms of the s- , t- and u-channel colour-correlation operators [88]

$$\begin{aligned} \mathbf{T}_s^2 &= (\mathbf{T}_1 + \mathbf{T}_2)^2 = (\mathbf{T}_3 + \mathbf{T}_4)^2 \ , \\ \mathbf{T}_t^2 &= (\mathbf{T}_1 + \mathbf{T}_3)^2 = (\mathbf{T}_2 + \mathbf{T}_4)^2 \ , \\ \mathbf{T}_u^2 &= (\mathbf{T}_2 + \mathbf{T}_3)^2 = (\mathbf{T}_1 + \mathbf{T}_4)^2 \ . \end{aligned} \quad (4.38)$$

They are linearly related by colour conservation:

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=4}^{4} \mathbf{T}_i^2 = \sum_{i=4}^{4} C_{a_i} \, . \quad (4.39)$$

Note also that $\mathbf{\Gamma}^{(1)}(r)$ and, more generally, $\mathbf{\Gamma}(\alpha_{\rm S}; r)$ depend on the kinematical (angular) variable r, at variance with the kernels $A_a^{\rm th}(\alpha_{\rm S})$ and $B_a^{\rm th}(\alpha_{\rm S})$ (which are independent of the kinematics) of Δ_{a_i,N_i} and J_{a_4,N_4} in Eqs. (3.28) and (3.47).

The colour-space amplitude $|\mathcal{M}_H\rangle$ depends on the flavour, colour and kinematical variables of the scattering process $a_1 a_2 \to a_3 a_4$ in Eq. (4.13), and it is independent of the Mellin moment N. It embodies the residual terms

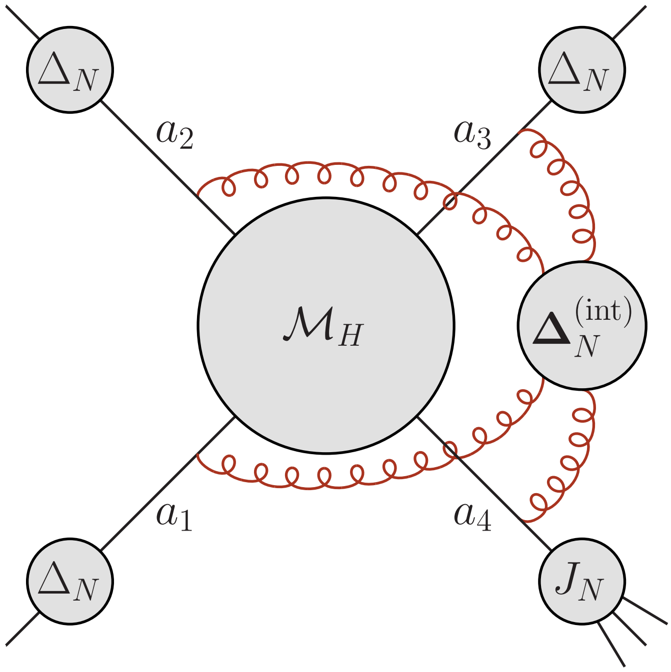

Figure 4.1: Pictorial representation of the resummed formula in Eq. (4.30).

of Σ_N^{res} that are constant, i.e. of $\mathcal{O}(1)$, and not logarithmically enhanced in the large-N limit. Its all-order perturbative structure is analogous to the structure of the scattering amplitude $|\mathcal{M}\rangle$ in Eq. (4.10). We write

$$|\mathcal{M}_H\rangle = \alpha_S(\mu_R^2)\left[|\mathcal{M}^{(0)}\rangle + \sum_{n=1}^{\infty}\left(\frac{\alpha_S(\mu_R^2)}{2\pi}\right)^n |\mathcal{M}_H^{(n)}(\mu_R)\rangle\right], \tag{4.40}$$

where we have omitted the explicit reference to the parton indices $a_1a_2a_3a_4$. At the lowest order, $\mathcal{M}_H$ exactly coincides with the Born-level scattering amplitude $\mathcal{M}^{(0)}$. The analogy between $\mathcal{M}_H$ and $\mathcal{M}$ persists at higher orders, since $\mathcal{M}_H$ also refers to the elastic scattering $a_1a_2 \to a_3a_4$ and it can be regarded as the 'hard' component of the virtual contributions to the renormalised scattering amplitude $\mathcal{M}$. The amplitude $|\mathcal{M}_H\rangle$ is obtained from $|\mathcal{M}\rangle$ by removing its IR divergences and a definite amount of IR finite terms. The terms that are removed from $|\mathcal{M}\rangle$ originate from the soft real emission contributions to the cross section. Therefore, they specifically depend on the one-parton inclusive cross section. $|\mathcal{M}_H\rangle$ is an observable-dependent quantity.

The first-order term $\mathcal{M}_H^{(1)}$ of $\mathcal{M}_H$ can be obtained from the results of our NLO calculation of the partonic cross section near threshold. We consider the NLO result in Eqs. (4.17) and (4.18), after the change of variables of Eq. (4.21). Then we compute its N-moments with respect to x_ω and, in the limit $N \to \infty$, we compare this result with the perturbative $\mathcal{O}(\alpha_S)$ expansion of the resummation formula in Eq. (4.30). From the comparison we cross-check that the structure and the coefficients of the double- and single-logarithmic terms do agree and we extract $|\mathcal{M}_H^{(1)}\rangle$ in explicit form, which reads

$$|\mathcal{M}_H^{(1)}\rangle = |\mathcal{M}^{(1)}\rangle - \mathbf{I}_H^{(1)}\, |\mathcal{M}^{(0)}\rangle\,. \tag{4.41}$$

The amplitude $|\mathcal{M}^{(1)}\rangle$ is the full one-loop scattering amplitude in Eqs. (4.10) and (4.14). The colour-space operator $\mathbf{I}_H^{(1)}$ has the following explicit form:

$$\begin{aligned}
\mathbf{I}_H^{(1)} = \mathbf{I}_{\text{sing}}^{(1)} &+ \frac{\pi^2}{4}\left(\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2 + \frac{4}{3}\mathbf{T}_4^2\right) + \frac{1}{2}\sum_{i=1}^{3}\gamma_{a_i}\ln\frac{\mu_{Fi}^2}{p_T^2} \\
&- \frac{1}{2}\ln(1+r)\ln\frac{1+r}{r}\left(\mathbf{T}_1^2 + \mathbf{T}_2^2 - 3\mathbf{T}_3^2 + \mathbf{T}_4^2\right) \\
&- \mathbf{T}_t^2\left(\frac{\pi^2}{2} + \frac{1}{2}\ln^2(1+r) + \ln(1+r)\ln\frac{1+r}{r}\right) \\
&- \mathbf{T}_u^2\left(\frac{\pi^2}{2} + \frac{1}{2}\ln^2\frac{1+r}{r} + \ln(1+r)\ln\frac{1+r}{r}\right) - \frac{1}{2}K_{a_4} &\quad (4.42)\\
&\equiv \mathbf{I}_{\text{sing}}^{(1)} + \mathbf{I}_H^{(1)\,\text{fin}}\,. &\quad (4.43)
\end{aligned}$$

The operator $\mathbf{I}_{\text{sing}}^{(1)}$ is defined in Eq. (4.15) and the flavour coefficients K_a are given in Eq. (4.20). Eq. (4.41) can be rewritten in terms of $\mathcal{M}^{(1)\,\text{fin}}$ in Eq. (4.14) and of the IR-finite part of $\mathbf{I}_H^{(1)}$, as defined by Eq. (4.43). We have

$$|\mathcal{M}_H^{(1)}\rangle = |\mathcal{M}^{(1)\,\text{fin}}\rangle - \mathbf{I}_H^{(1)\,\text{fin}}\, |\mathcal{M}^{(0)}\rangle\,, \tag{4.44}$$

which explicitly shows that the one-loop hard-virtual amplitude $\mathcal{M}_H^{(1)}$ is IR finite.

We note that the colour-space factorisation form of our NLO result in Eqs. (4.17) and (4.18) is essential to obtain the *amplitude* $|\mathcal{M}_H^{(1)}\rangle$. The determination of the amplitude, rather than the colour-summed interference '$\langle\mathcal{M}^{(0)}|\mathcal{M}_H^{(1)}\rangle$ + c.c.', is important for QCD predictions beyond the NLO.

4.3 Extension to NNLL accuracy

The factorised structure of $\langle\mathcal{M}_H|\boldsymbol{\Delta}_N^{(\text{int})}|\mathcal{M}_H\rangle$ in colour space entails colour interferences between $\boldsymbol{\Delta}_N^{(\text{int})}$ and $|\mathcal{M}_H\rangle$. The colour interference effects start to contribute at $\mathcal{O}(\alpha_S^2)$. The dominant logarithmic terms in $\mathbf{\Delta}_N^{(\mathrm{int})}$ are of $\mathcal{O}(\alpha_S^n \ln^n N)$, and we can consider the following approximation:

$$\langle \mathcal{M}_H | \mathbf{\Delta}_N^{(\mathrm{int})} | \mathcal{M}_H \rangle \to \langle \mathcal{M}^{(0)} | \mathbf{\Delta}_N^{(\mathrm{int})} | \mathcal{M}^{(0)} \rangle \frac{|\mathcal{M}_H|^2}{|\mathcal{M}^{(0)}|^2} + \mathcal{O}\left(\alpha_S (\alpha_S \ln N)^n\right) , \tag{4.45}$$

which shows that the colour interference effects can be neglected up to $\mathcal{O}((\alpha_S \ln N)^n)$. Starting from $\mathcal{O}\left(\alpha_S(\alpha_S \ln N)^n\right)$, the colour interference effects are relevant. In particular, $\langle \mathcal{M}_H | \mathbf{\Delta}_N^{(\mathrm{int})} | \mathcal{M}_H \rangle$ leads to the second-order contribution

$$-\frac{1}{2}\left(\frac{\alpha_S}{\pi}\right)^2 \ln N \Big\{ + 2 \langle \mathcal{M}^{(0)} | \left(\mathbf{\Gamma}^{(2)} + \mathbf{\Gamma}^{(2)\dagger} \right) | \mathcal{M}^{(0)} \rangle + \left[\langle \mathcal{M}^{(0)} | \left(\mathbf{\Gamma}^{(1)} + \mathbf{\Gamma}^{(1)\dagger} \right) | \mathcal{M}_H^{(1)} \rangle + \mathrm{c.c.} \right] \Big\} , \tag{4.46}$$

that is incorrectly approximated by neglecting colour interferences as in the right-hand side of Eq. (4.45). Using the approximation in Eq. (4.45), the last term in the curly bracket of Eq. (4.46) would be replaced by

$$\langle \mathcal{M}^{(0)} | \left(\mathbf{\Gamma}^{(1)} + \mathbf{\Gamma}^{(1)\dagger} \right) | \mathcal{M}^{(0)} \rangle \frac{\langle \mathcal{M}^{(0)} | \mathcal{M}_H^{(1)} \rangle + \mathrm{c.c.}}{|\mathcal{M}^{(0)}|^2} . \tag{4.47}$$

The expression in Eq. (4.46) explicitly shows that the second-order anomalous dimension $\mathbf{\Gamma}^{(2)}$ contributes at the same level of logarithmic accuracy as the colour interference between $\mathbf{\Gamma}^{(1)}$ and $|\mathcal{M}_H^{(1)}\rangle$.

As discussed in the comments to Eq. (4.46), thanks to our factorised formula we can extend the soft-gluon resummation beyond the next-to-leading logarithmic accuracy. The complete determination of the NNLL terms in $\mathcal{G}_I$ requires the explicit inclusion of higher-order terms in the kernels of the radiative functions $\Delta_{a,N}, J_{a,N}, \mathbf{\Delta}_N^{(\mathrm{int})}$. The soft-collinear NNLL terms of Eqs. (3.28) and (3.47) are controlled by the coefficient $A_a^{\mathrm{th}(3)}$ in Eq. (3.24). It is the third-order coefficient of the soft part of the Altarelli-Parisi splitting function $P_{aa}(z, \alpha_S)$ and it is already known [71]. The collinear (non-soft) NNLL terms of Eq. (3.47) are controlled by the coefficient $B_a^{\mathrm{th}(2)}$ in Eq. (3.45). This term could be extracted from NNLL computations of related processes, such as DIS [72, 75, 89] and direct-photon production [54]. As regards large-angle soft-gluon emission, the NNLL terms are generated by the second-order anomalous dimension $\mathbf{\Gamma}^{(2)}(r)$ in Eq. (4.35) and by the colour interference (see Eq. (4.46)) between the first-order anomalous dimension $\mathbf{\Gamma}^{(1)}$ and the one-loop hard-virtual amplitude $|\mathcal{M}_H^{(1)}\rangle$. Since $\mathbf{\Gamma}^{(1)}$ and $|\mathcal{M}_H^{(1)}\rangle$ are known, also their interference is known. In the colour-diagonalised expression of

Eq. (4.49), the interference is taken into account by the correlated dependence on I between $\widetilde{C}_I^{(1)}$ and $\Gamma_I^{(1)}$. Finally, the bulk of the contributions to $\mathbf{\Gamma}^{(2)}(r)$ is expected [37,44,67,90] to be proportional to $\mathbf{\Gamma}^{(1)}(r)$. It is obtained by inserting the simple rescaling

$$\alpha_{\rm S} \to \alpha_{\rm S} \left[1 + \frac{\alpha_{\rm S}}{\pi} \frac{1}{2} K\right] \tag{4.48}$$

in the expression of $\mathbf{\Gamma}$ at $\mathcal{O}(\alpha_{\rm S})$. The coefficient K is given in Eq. (3.25).

We conclude by showing how our colour-space resummation formula can be cast to an explicit (scalar) formula like that of Eq. (3.6). The all-order structure of Eqs. (4.30), (3.28), (3.47) and (4.34) leads to the resummation of the $\ln N$ terms in exponentiated form. However, in Eq. (4.34) 'exponentiation' has a formal meaning, since it refers to the formal exponentiation of matrices. The anomalous dimension matrix $\mathbf{\Gamma}(\alpha_{\rm S}; r)$ must be first diagonalised in colour space [41, 88]. After that, the resummed radiative function $\Sigma_N^{\rm res}$ of Eq. (4.30) can be written in the customary exponential form

$$\begin{aligned} \Sigma^{\rm res}_{a_1a_2\to a_3a_4,N}(r;p_T^2,\mu_F,\mu_f) = \sum_I \; & \widetilde{C}_{I,a_1a_2a_3a_4}(\alpha_{\rm S}(p_T^2), r; p_T^2, \mu_F, \mu_f) \\ & \times \exp\Big\{\mathcal{G}_{I,a_1a_2a_3a_4}(\alpha_{\rm S}(p_T^2), \ln N, r; p_T^2, \mu_F, \mu_f)\Big\} + \mathcal{O}\left(\frac{1}{N}\right) , \end{aligned} \tag{4.49}$$

where the index I labels the colour-space eigenstates $|I(\alpha_{\rm S}; r)\rangle$ of $\mathbf{\Gamma}(\alpha_{\rm S}; r)$, and $\widetilde{C}$ and $\mathcal{G}$ are functions (they are not colour matrices). These functions are renormalisation group invariant, and their dependence on μ_R arises by writing $\alpha_{\rm S}(p_T^2)$ as a function of $\alpha_{\rm S}(\mu_R^2)$ and $\ln(p_T^2/\mu_R^2)$ (as in customary perturbative calculations).

The exponent function $\mathcal{G}_I$ includes all the $\ln N$ terms, and it can consistently be expanded in LL terms of $\mathcal{O}(\alpha_{\rm S}^n \ln^{n+1} N)$, NLL terms of $\mathcal{O}(\alpha_{\rm S}^n \ln^n N)$, NNLL terms of $\mathcal{O}(\alpha_{\rm S}(\alpha_{\rm S} \ln N)^n)$, and so forth, corresponding to various terms in the r.h.s. of Eq. (3.6). The function $\widetilde{C}_I$ does not depend on N, since it includes all the terms that are constant (i.e., of $\mathcal{O}(1)$) in the large-N limit. The LL terms of $\mathcal{G}_I$ are actually independent of the colour eigenstate I and are controlled by the perturbative coefficient $A_a^{\rm th(1)}$ in Eq. (3.24). The NLL terms of $\mathcal{G}_I$ are fully determined by $A_a^{\rm th(2)}$ of Eq. (3.26), $B_a^{\rm th(1)}$ of Eq. (3.45) and the eigenvalues $\Gamma_I^{(1)}(r)$ of $\mathbf{\Gamma}^{(1)}$ in Eq. (4.36). The Born-level contribution to the function $\widetilde{C}_I$ depends on $|\langle I|\mathcal{M}^{(0)}\rangle|^2$. The first-order term $\widetilde{C}_I^{(1)}$ depends on the colour interference $(\langle \mathcal{M}^{(0)}|I\rangle\langle I|\mathcal{M}_H^{(1)}\rangle + \text{c.c.})$', which is computable from the explicit expression of $|\mathcal{M}_H^{(1)}\rangle$, given by Eqs. (4.41) and (4.42).

Chapter 5

Transverse-momentum Resummation

The transverse-momentum ($q_{\rm T}$) spectrum of high-mass systems produced in hadronic collisions is sensitive to soft-gluon effects, in the region of vanishing $q_{\rm T}$. If the final-state system is kinematically constrained to have a small transverse momentum, the emission of accompanying radiation is strongly inhibited. In this case only soft and collinear partons can be produced in the final state, other than the high-mass system. For each radiated parton, the corresponding cross section contains a single- or double-logarithmic term, which is singular in the $q_{\rm T} \to 0$ limit. This is analogous to what happens by approaching the partonic-threshold region (see Chapter 3). Here we focus on the production of a massive (scalar or vector) boson by the hard-scattering of partons and we discuss the all-order transverse-momentum resummation, following the BCDG resummation formalism [29]. The same resummed formula applies to the transverse-momentum distribution of generic *colourless* high-mass systems produced in hadron-hadron collisions, such as vector bosons and lepton pairs, as long as the invariant mass of the system is measured. The colour algebra for these processes is simple. It factorises into the Casimir operators of the initial-state partons, while the colour-correlation terms due to the exchange of non-collinear gluons between the two partons cancel out in the total amplitude, in the soft limit. The large logarithmic terms are therefore fully resummed by the radiative factors of the colliding partons, the Sudakov form factors of the CSS [28] formula. As shown in [91, 92], the form factors of the conventional resummation formalism are actually process-dependent. The process dependence can however be collected in one single coefficient [93] and it is possible to define universal quark and gluon form factors. The advantage of this approach is that the large logarithmic contributions can be exponentiated in a process-independent form, and the QCD infrared radiation is treated (and understood) in a universal way. The transverse-momentum distribution for different processes can thus be described by the the same factorised formula. The effects of non-soft virtual corrections are fixed by hard-matching, i.e. by comparison with the known fixed-order results. In Chapter 6 we present an extension of the formalism in [29] to describe the production of strongly interacting particles and to include non-trivial colour-correlation effects due to the exchange of large-angle soft-gluons in multiparton scatterings.

As anticipated in Section 2.2, the resummation has to be carried out in impact-parameter space, in order to correctly factorise the transverse momentum constraint for multiparton emission. The impact-parameter was first introduced in [26], where the (inverse) Sudakov parameter is b^2Q^2 and the leading terms are $\alpha_S^n \ln(b^2Q^2)^{2n}$ in the large b limit. The same techniques were applied [94,95] to a similar process, e^+e^- annihilation into hadrons, and they were improved to include resummation of subleading single-logarithmic terms, first in e^+e^- annihilation [96], then in the Drell-Yan process [97].

5.1 The transverse-momentum cross section at NLO

We consider the hadronic collision between two hadrons h_1 and h_2 with momenta P_1 and P_2, producing a (composite) final state F of measured invariant mass M, transverse momentum $\mathbf{q}_\perp$ and rapidity y,

$$h_1(P_1) + h_2(P_2) \to F(\mathbf{q}_\perp, y, M, \mathbf{\Omega}) + X \,. \tag{5.1}$$

The system F is a colourless system of one or more non-QCD partons, such as lepton pairs, vector bosons and Higgs bosons and X denotes all the accompanying final-state radiation. $\mathbf{\Omega}$ represents a set of additional variables that controls the internal kinematics of system F. We define the hadronic scaling variables x_1, x_2 such that $M^2 = x_1x_2s$ and $y = 1/2\ln(x_1/x_2)$, where $s = (P_1+P_2)^2$ is the square of the centre-of-mass energy of hadronic collision. Explicitly

$$x_1 = \frac{M}{\sqrt{s}}\, e^{+y} \,, \qquad x_2 = \frac{M}{\sqrt{s}}\, e^{-y} \,. \tag{5.2}$$

The QCD factorisation theorem connects the cross section $d\sigma_{h_1h_2\to F+X}$ for the hadronic process in Eq. (5.3) to the cross section $d\hat{\sigma}_{a_1a_2\to F+X}$ for the partonic scattering process

$$a_1(p_1) + a_2(p_2) \to F(\mathbf{q}_\perp, \hat{y}, M, \mathbf{\Omega}) + X\,, \tag{5.3}$$

where p_1, p_2 are the four-momenta of the initial-state partons and $\hat{y}$ is the rapidity of the system F in the partonic center-of-mass frame. If ξ_1, ξ_2 are the longitudinal-momentum fractions of the colliding hadrons carried by the initial-state partons, such that $\xi_i \in (x_i, 1)$ for $i = 1, 2$, then the relation between the partonic and hadronic momenta is given by $p_i = \xi_i P_i$. In the massless-parton approximation, the partonic centre-of-mass energy $\hat{s} = (p_1 + p_2)^2$ and the rapidity $\hat{y}$ are fully determined by ξ_1, ξ_2 and by the hadronic variables s, y:

$$\hat{s} = \xi_1 \xi_2 s\,, \qquad \hat{y} = y - \frac{1}{2} \ln \frac{\xi_1}{\xi_2}\,. \tag{5.4}$$

It is also convenient to define the partonic scaling variables z_1, z_2 such that $M^2 = z_1 z_2 \hat{s}$ and $\hat{y} = 1/2 \ln(z_1/z_2)$. Explicitly

$$z_1 = \frac{M}{\sqrt{\hat{s}}}\, e^{+\hat{y}}\,, \qquad z_2 = \frac{M}{\sqrt{\hat{s}}}\, e^{-\hat{y}}\,. \tag{5.5}$$

We can now define the fully-differential cross section

$$\begin{aligned} \frac{d\sigma_{h_1 h_2 \to F+X}(s; \mathbf{q}_\perp, y, M, \mathbf{\Omega})}{d^2\mathbf{q}_\perp\, dM^2\, dy\, d\mathbf{\Omega}} &= \frac{1}{s} \sum_{a_1, a_2} \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} \\ \times\, f_{a_1/h_1}(x_1/z_1, \mu_F^2)\, f_{a_2/h_2}(x_2/z_2, \mu_F^2)\, & \frac{d\hat{\sigma}_{a_1 a_2 \to F+X}(\mathbf{q}_\perp, z_1, z_2; M, \mathbf{\Omega}; \mu_F)}{d^2\mathbf{q}_\perp\, dz_1\, dz_2\, d\mathbf{\Omega}}\,. \end{aligned} \tag{5.6}$$

The partonic cross-section $d\hat{\sigma}_{a_1 a_2 \to F+X}$ in the r.h.s. of Eq. (5.6) is computable in perturbative QCD as a power expansion in α_S. At the parton level, in the lowest-order approximation, the system F is produced with no accompanying final-state radiation, at zero q_T. The Born-level contribution to the cross-section $d\hat{\sigma}_{a_1 a_2 \to F+X}$ is then proportional to $\delta(q_T^2)\delta(1 - z_1)\delta(1 - z_2)$ and is due to the elastic scattering processes

$$c(p_1) + \bar{c}(p_2) \to F(\mathbf{q}_\perp = 0, \hat{y} = 0, M, \mathbf{\Omega})\,, \tag{5.7}$$

where the initial-state partons are quark or gluon pairs ($c = q, \bar{q}, g$). The higher-order contributions contain logarithmic terms $\ln(M^2/q_T^2)$, singular in the limit $q_T \to 0$. These terms must be resummed to all orders. We can decompose the partonic cross-section in a *singular* and a *regular* part,

$$d\hat{\sigma}_{a_1 a_2 \to F+X} = d\hat{\sigma}^{(\mathrm{sing})}_{a_1 a_2 \to F+X} + d\hat{\sigma}^{(\mathrm{reg})}_{a_1 a_2 \to F+X}\,, \tag{5.8}$$

the first containing all the terms that are enhanced in the low-q_{T} region, the other free of such terms. The $d\hat{\sigma}^{(\mathrm{reg})}_{a_1a_2\to F+X}$ component can be calculated at fixed orders, while the $d\hat{\sigma}^{(\mathrm{sing})}_{a_1a_2\to F+X}$ component must be replaced by its resummed version $d\hat{\sigma}^{(\mathrm{res})}_{a_1a_2\to F+X}$. This is the same procedure sketched by Eqs. (3.3–3.6). In the following we ignore the finite component, suppressed at low q_{T}, to focus on the singular terms.

The decomposition of Eq. (5.8) can be straightforwardly applied to the hadronic cross section in Eq. (5.6). The singular component $d\sigma^{(\mathrm{sing})}_{h_1h_2\to F+X}$ is then equal to the sum, over all the possible flavours a_1, a_2, of the convolutions between the parton densities $f_{a_1/h_1}, f_{a_2/h_2}$ and the partonic cross-sections $d\hat{\sigma}^{(\mathrm{sing})}_{a_1a_2\to F+X}$. We start by considering the perturbative QCD corrections to the Born-level amplitude of relative order $\mathcal{O}(\alpha_{\mathrm{S}})$. In the low-$q_{\mathrm{T}}$ region, the dominant contributions come from the full (hard and soft) one-loop corrections to the elastic scattering processes of Eq. (5.7) and from the tree-level contribution of the inelastic scattering processes $a_1a_2 \to F + X$ of Eq. (5.3), where X is either a soft gluon or a hard parton collinear to one of the initial-state partons a_1, a_2. The enhanced logarithmic terms can be calculated by following a procedure similar to that of Chapter 3. The total virtual correction is decomposed in a hard and a soft contribution. The former is finite and process dependent and it is equal to an hard-vitual term $H^{(1)}$ proportional to $\delta(q_{\mathrm{T}}^2)\delta(z_1)\delta(z_2)$, i.e. to the Born-level kinematics (cfr. Eq. (5.9)). This term does not contribute with enhanced logarithmic terms and it does not need to be resummed, in analogy to the C^{DIS} in Eq. (3.12) and C^{DY} in Eq. (3.33). The latter, the soft-virtual term, is divergent and universal: it contains IR poles of QCD, fixed by unitarity [66, 67, 85, 98]. The sum of this contribution with the collinear counterterms (PDFs) and the real corrections is finite and amounts to enhanced logarithmic terms, constant terms and power-suppressed terms.

We give the explicit definition of the hard-virtual term in Section 5.3, to focus in the following on the real corrections. These can be calculated, at NLO, by employing the eikonal approximation of the single-emission matrix elements, complemented by the full Altarelli-Parisi splitting functions, in order to include non-soft collinear radiation. Indeed, at variance with the partonic-threshold limit, where initial-state and registered partons can contribute only with soft-gluon emissions, the low-q_{T} limit allows both soft and collinear-hard emissions from initial-state partons. It follows that the eikonal approximation would not include all the IR poles and singular terms. It is important to understand the different kinematics of the partonic-threshold limit and the vanishing transverse-momentum limit. If we consider Drell-Yan production or DIS, the enhanced terms are produced in the near-threshold region, defined by the $z \to 1$ limit: the parameter $z = M^2/\hat{s}$ does not discriminate between soft and hard-collinear emissions. Now the enhanced terms are produced in the $q_\mathrm{T} \to 0$ limit and q_T does not discriminate between soft and hard-collinear emissions neither. Nevertheless, the partonic cross section in the r.h.s. of Eq. (5.6) is fully-differential with respect to the total four-momentum q^μ of the final-state system F, at fixed center-of-mass energy $\hat{s}$ of the partonic collision. Namely, also the invariant mass M^2 and the rapidity $\hat{y}$ of system F are known, besides its transverse-momentum $\mathbf{q}_\perp$, and this additional information is enough to discriminate between soft and hard-collinear emissions. From $M^2/\hat{s}$ we can control the partonic-threshold limit: it corresponds, at fixed M^2, to the limit $z_1 z_2 \to 1$, which is reached iff $z_1 \to 1$ and $z_2 \to 1$. From $\hat{y}$ it is moreover possible to distinguish between the two initial-state partons: the parameter z_i ($i = 1, 2$) describes the softness of the emission from parton a_i and the limit $z_i \to 1$ corresponds to soft-gluon emission from parton a_i. In the low-q_T limit the parameter z_i can be equal to 1, corresponding to soft (or no) emission from parton a_i or smaller than 1, corresponding to the emission of, at least, one non-soft parton collinear to parton a_i.

The singular NLO cross section at the factorisation scale μ_F reads:

$$\begin{aligned}\frac{d\hat{\sigma}^{\mathrm{sing}}_{a_1 a_2 \to F+X}}{d^2\mathbf{q}_\perp\, dz_1\, dz_2\, d\mathbf{\Omega}} = \frac{M^2}{\pi} \sum_{c=q,\bar{q},g} [d\sigma^{(0)}_{c\bar{c},F}] \Big\{ & \delta(q_\mathrm{T}^2)\, \delta(1-z_1)\, \delta(1-z_2)\, \delta_{ca_1}\delta_{\bar{c}a_2} \\ & + \frac{\alpha_\mathrm{S}}{\pi} H_c^{F(1)}(M,\mathbf{\Omega})\, \delta(q_\mathrm{T}^2)\, \delta(1-z_1)\, \delta(1-z_2)\, \delta_{ca_1}\delta_{\bar{c}a_2} \\ & + \frac{\alpha_\mathrm{S}}{\pi} W^{\mathrm{DY}(1)}_{c\bar{c};a_1 a_2}(q_\mathrm{T}^2, z_1, z_2; M; \mu_F) + \mathcal{O}(\alpha_\mathrm{S}^2) \Big\}\,, \end{aligned} \tag{5.9}$$

where

$$[d\sigma^{(0)}_{c\bar{c},F}] = \frac{d\hat{\sigma}^{(0)}_{c\bar{c}\to F}(M,\mathbf{\Omega})}{M^2\, d\mathbf{\Omega}}\,. \tag{5.10}$$

The first line in the r.h.s. of Eq. (5.9) is the LO contribution and the second line is the NLO hard-virtual term, with Born-level kinematic ($q_\mathrm{T} = 0$, $z_1 = z_2 = 1$). The third line corresponds to the soft corrections from virtual and real emission, with single-emission probability $W^{\mathrm{DY}(1)}_{c\bar{c};a_1a_2}$ equal to

$$W^{\mathrm{DY}(1)}_{c\bar{c};a_1a_2} = \left[C_c \left(\frac{\ln(M^2/q_{\mathrm{T}}^2)}{q_{\mathrm{T}}^2} \right)_+ - \gamma_c \left(\frac{1}{q_{\mathrm{T}}^2} \right)_+ \right] \delta(1-z_1)\, \delta(1-z_2)\, \delta_{ca_1} \delta_{\bar{c}a_2}$$
$$+ \left[\left(\frac{1}{q_{\mathrm{T}}^2} \right)_+ + \ln \frac{M^2}{\mu_F^2}\, \delta(q_{\mathrm{T}}^2) \right] \left(P^{(1)}_{ca_1}(z_1)\, \delta(1-z_2)\, \delta_{\bar{c}a_2} + P^{(1)}_{\bar{c}a_2}(z_2)\, \delta(1-z_1)\, \delta_{ca_1} \right)$$
$$- \delta(q_{\mathrm{T}}^2) \left(\hat{P}^{\epsilon}_{ca_1}(z_1)\, \delta(1-z_2)\, \delta_{\bar{c}a_2} + \hat{P}^{\epsilon}_{\bar{c}a_2}(z_2)\, \delta(1-z_1)\, \delta_{ca_1} \right) . \tag{5.11}$$

The first line in the r.h.s. of Eq. (5.11) corresponds to soft-gluon emission and contains double-logarithmic terms, from soft and collinear divergencies, while the second one contains only single-logarithmic terms and the third one is free of enhanced terms. The coefficient C_c is the Casimir operator of parton c ($C_q = C_{\bar{q}} = C_F$, $C_g = C_A$) and the flavour-dependent coeffcient γ_c is defined in Eq. (3.42). The second and third lines have an hard-collinear origin: they are defined in terms of Altarelli-Parisi splitting probabilities of the initial state partons and they admit contribution far from the partonic-threshold region $z_1 z_2 = 1$. Here we have defined the ϵ-dependent unregularised splitting functions

$$\hat{P}_{a_ib_i}(z, \epsilon; \alpha_{\mathrm{S}}) = \frac{\alpha_{\mathrm{S}}}{\pi} \left(\hat{P}^{(1)}_{a_ib_i}(z) + \epsilon\, \hat{P}^{\epsilon}_{a_ib_i}(z) + \mathcal{O}(\epsilon^2) \right) + \mathcal{O}(\alpha_{\mathrm{S}}^2) , \tag{5.12}$$

with $\mathcal{O}(\epsilon)$ lowest-order coefficients equal to

$$\hat{P}^{\epsilon}_{qq}(z) = \frac{1}{2} C_F (1-z) , \tag{5.13}$$
$$\hat{P}^{\epsilon}_{gq}(z) = \frac{1}{2} C_F\, z , \tag{5.14}$$
$$\hat{P}^{\epsilon}_{qg}(z) = \frac{1}{2} z(1-z) , \tag{5.15}$$
$$\hat{P}^{\epsilon}_{gg}(z) = \hat{P}^{\epsilon}_{q\bar{q}}(z) = \hat{P}^{\epsilon}_{qq'}(z) = \hat{P}^{\epsilon}_{q\bar{q}'}(z) = 0 . \tag{5.16}$$

The labels q and q' denote quarks with different flavours. We note that the $\hat{P}_{a_ib_i}(z, \epsilon; \alpha_{\mathrm{S}})$ functions are *unpolarised* splitting probabilities, independent of the spin of the radiated partons. This simplification holds only in the $q\bar{q}$ channel (see Section 5.4) and is lost when the final-state system F has a non-trivial colour structure and is produced through gluon fusion.

In order to proceed with the exponentiation of the logarithmic terms, a Fourier transform to the impact-parameter space is required (see Section 2.2). The impact-parameter vector $\mathbf{b}$ is conjugate to $\mathbf{q}_\perp$ while its norm, the parameter $b^2 = \mathbf{b}^2$, is conjugate to the transverse-momentum $q_\mathrm{T}^2 = \mathbf{q}_\perp^2$. The partonic cross section in $\mathbf{b}$-space reads

$$\frac{d\hat{\sigma}_{a_1 a_2 \to F+X}(\mathbf{q}_\perp; \ldots)}{d^2\mathbf{q}_\perp \, dz_1 \, dz_2 \, d\mathbf{\Omega}} = \int \frac{d^2\mathbf{b}}{(2\pi)^2} \, e^{i\mathbf{b}\cdot\mathbf{q}_\perp} \frac{d\hat{\sigma}_{a_1 a_2 \to F+X}(\mathbf{b}\, ; \ldots)}{d^2\mathbf{b} \, dz_1 \, dz_2 \, d\mathbf{\Omega}} \,. \tag{5.17}$$

The same definition holds for each $\mathbf{q}_\perp$-dependent function in Eq. (5.11). In order to cast Eq. (5.11) to this form we need to Fourier-transform the q_T-distributions. To this purpose we employ an integral representation of the plus-distributions in terms of delta-distributions and we transform the latter to b-space:

$$\begin{aligned}
\frac{1}{\pi}\left(\frac{\ln^n(M^2/q_\mathrm{T}^2)}{q_\mathrm{T}^2}\right)_+ &= \frac{1}{\pi}\int d^2\mathbf{k}_\perp \frac{\ln^n(M^2/k_\mathrm{T}^2)}{k_\mathrm{T}^2}\left(\delta^{(2)}(\mathbf{q}_\perp - \mathbf{k}_\perp) - \delta^{(2)}(\mathbf{q}_\perp)\right) \\
&= \int \frac{d^2\mathbf{b}}{(2\pi)^2} \, e^{i\mathbf{b}\cdot\mathbf{q}_\perp} \frac{1}{\pi}\int d^2\mathbf{k}_\perp \frac{\ln^n(M^2/k_\mathrm{T}^2)}{k_\mathrm{T}^2}\left(e^{-i\mathbf{b}\cdot\mathbf{k}_\perp} - 1\right) \\
&= \int \frac{d^2\mathbf{b}}{(2\pi)^2} \, e^{i\mathbf{b}\cdot\mathbf{q}_\perp} \int_0^{M^2} dk_\mathrm{T}^2 \, \frac{\ln^n(M^2/k_\mathrm{T}^2)}{k_\mathrm{T}^2} \left(J_0(b\, k_\mathrm{T}) - 1\right) \,.
\end{aligned} \tag{5.18}$$

The third line of Eq. (5.18) follows from the integral over the azimuthal degrees-of-freedom of the transverse momentum $\mathbf{k}_\perp$: in this case, with only two QCD partons involved in the hard scattering, the cross-section is actually independent of the azimuthal degree-of-freedom, the Fourier transform simplifies to a Bessel transform and the transformed cross-section only depends on the parameter b. The function J_0 is the 0th-order Bessel function. The region $q_\mathrm{T} \ll M$ corresponds to the region $Mb \gg 1$ in the conjugate space. To evaluate the integrals in Eq. (5.18) we can exploit an approximation similar to Eq. (2.45). The logarithmic contributions at large b can indeed be obtained by observing that the Bessel function $J_0(b\, k_\mathrm{T})$ quickly oscillates when $b\, k_\mathrm{T} \sim 1$. As a consequence we can simply set

$$J_0(b\, k_\mathrm{T}) \sim \Theta(b_0^2/b^2 - k_\mathrm{T}^2) \,, \tag{5.19}$$

where the choice $b_0 = 2e^{-\gamma_E}$ ($\gamma_E = 0.5772\ldots$ is the Euler number) allows us to control the large logarithmic terms up to NLL accuracy. We get the simple expression

$$\frac{1}{\pi}\left(\frac{\ln^n(M^2/q_\mathrm{T}^2)}{q_\mathrm{T}^2}\right)_+ = -\int \frac{d^2\mathbf{b}}{(2\pi)^2} \, e^{i\mathbf{b}\cdot\mathbf{q}_\perp} \int_{b_0^2/b^2}^{M^2} dk_\mathrm{T}^2 \, \frac{\ln^n(M^2/k_\mathrm{T}^2)}{k_\mathrm{T}^2} \,. \tag{5.20}$$

We can now easily transform Eqs. (5.9) and (5.11) to the conjugate space. The first two lines in the r.h.s. of Eq. (5.9) are constant in b-space, as well as the third line in the r.h.s. of Eq. (5.11). The first and second lines in the r.h.s. of Eq. (5.11) produce the large logarithmic terms. The former is trasformed to

$$-\int_{b_0^2/b^2}^{M^2} \frac{dk_{\mathrm{T}}^2}{k_{\mathrm{T}}^2} \left[C_c \ln \frac{M^2}{k_{\mathrm{T}}^2} - \gamma_c \right] , \tag{5.21}$$

and the latter to

$$\int_{\mu_F^2}^{b_0^2/b^2} \frac{dk_{\mathrm{T}}^2}{k_{\mathrm{T}}^2} \left(P_{ca_1}^{(1)}(z_1)\, \delta(1-z_2)\, \delta_{\bar{c}a_2} + P_{\bar{c}a_2}^{(1)}(z_2)\, \delta(1-z_1)\, \delta_{ca_1} \right) . \tag{5.22}$$

The expression in Eq. (5.22) is manifestly zero at the factorisation scale $\mu_F^2 = b^2/b_0^2$. We conclude that the correct choice for the factorisation scale is a running scale depending on (the inverse of) the impact parameter b, rather than a fixed scale μ_F of the order of the hard scale M. This choice conveniently resums the large logarithmic terms in Eq. (5.22), by including them into the evolution of the PDFs from the soft scale $1/b$ to the hard scale μ_F (cfr. Eq. (5.33)). The expression in Eq. (5.22) indeed presents the hard-collinear structure typical of the DGLAP evolution equations, with either z_1 or z_2 smaller than 1, corresponding to non-soft emission from either parton a_1 or parton a_2.

With the choice $\mu_F^2 = b^2/b_0^2$ the convolution between the partonic cross-section and the PDFs is defined in the conjugate space and the hadronic cross section has the factorised form

$$\begin{aligned} \frac{d\sigma_{h_1h_2\to F+X}(s;\mathbf{q}_\perp,y,M,\mathbf{\Omega})}{d^2\mathbf{q}_\perp\, dM^2\, dy\, d\mathbf{\Omega}} &= \frac{M^2}{s} \sum_{c=q,\bar{q},g} [d\sigma_{c\bar{c},F}^{(0)}] \int \frac{d^2\mathbf{b}}{(2\pi)^2}\, e^{i\mathbf{b}\cdot\mathbf{q}_\perp} S_c(M,b) \\ \times \sum_{a_1,a_2} \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} &\left[H^F C_1 C_2 \right]_{c\bar{c};a_1a_2} f_{a_1/h_1}\!\left(\frac{x_1}{z_1}, \frac{b_0^2}{b^2} \right) f_{a_2/h_2}\!\left(\frac{x_2}{z_2}, \frac{b_0^2}{b^2} \right), \end{aligned} \tag{5.23}$$

where M^2/b^2 power-suppressed terms have been neglected. The symbolic factor $\left[H^F C_1 C_2\right]$ contains all the terms that are constant, i.e. not enhanced, in the large-b limit: the process-dependent hard-virtual term H_c^F and the *universal* functions C_{ca_1}, $C_{\bar{c}a_2}$, that describes unpolarized collinear radiation from the initial-state partons. These functions admit a fixed-order perturbative expansion in α_{S} and are free of large logarithmic terms. From the NLO result in Eqs. (5.9) and (5.11) it follows that

$$H_c^F(M,\mathbf{\Omega};\alpha_{\mathrm{S}}) = 1 + \frac{\alpha_{\mathrm{S}}}{\pi}\, H_c^{F(1)}(M,\mathbf{\Omega}) + \mathcal{O}(\alpha_{\mathrm{S}}^2)\,, \tag{5.24}$$

$$C_{ab}(z;\alpha_{\mathrm{S}}) = \delta(1-z)\,\delta_{ab} - \frac{\alpha_{\mathrm{S}}}{\pi}\, \hat{P}_{ab}^{\epsilon}(z) + \mathcal{O}(\alpha_{\mathrm{S}}^2)\,. \tag{5.25}$$

The $\mathcal{O}(\alpha_S)$ term of function $H_c^F(M, \alpha_S)$ is responsible for the hard-virtual term in the second line of Eq .(5.9) and the $\mathcal{O}(\alpha_S)$ term of function $C_{ab}(z; \alpha_S)$ produces the third line in the r.h.s. of Eq. (5.11). The function $S_c(M, b)$ contains the large logarithmic terms of the first line of Eq. (5.11), Fourier-transformed via the prescription in Eq. (5.20). It is defined, up to NLO, from the result in Eq. (5.21):

$$S_c^{(\mathrm{f.o.})}(M, b) = 1 - \frac{\alpha_S}{\pi} \int_{b_0^2/b^2}^{M^2} \frac{dk_T^2}{k_T^2} \left[C_c \ln \frac{M^2}{k_T^2} - \gamma_c \right] + \mathcal{O}(\alpha_S^2) \,. \tag{5.26}$$

The expression in Eq. (5.23), with the functions H_c^F, C_{ab} and S_c defined in Eqs. (5.24), (5.25) and (5.26) respectively, gives the NLO singular contribution to the fully-differential cross section of the hadronic scattering in Eq. (5.1). The derivation of the resummed cross section from the fixed-order result is discussed in the following section.

5.2 The resummation formula

The all-order resummed cross section is formally given by the same expression that defines the fixed-order cross section, in Eq. (5.23), where the function $S_c^{(\mathrm{f.o.})}$ in Eq. (5.26) is replaced by its resummed version. This function is analogous to the corresponding NLO results for the radiative-functions $\Delta_{c,N}$ and $J_{c,N}$, whose leading singular terms are resummed by simply exponentiating them (see Chapter 3), with the coupling α_S evaluated at the running scale k_T^2 [70]. Along the same lines, we promote the expression in Eq. (5.26) to the exponential

$$\begin{aligned} S_c(M, b) = \exp\Bigg\{ & - \int_{b_0^2/b^2}^{M^2} \frac{dk_T^2}{k_T^2} \frac{\alpha_S(k_T^2)}{\pi} \left[C_c \ln \frac{M^2}{k_T^2} - \gamma_c \right] \\ & + \mathcal{O}(\alpha_S \ln(M^2/b^2)) \Bigg\} \,, \end{aligned} \tag{5.27}$$

valid up to LL accuracy. The subleading terms can be included by employing the formal prescriptions

$$+\frac{\alpha_S}{\pi} C_c \to A_c(\alpha_S) \,, \tag{5.28}$$

$$-\frac{\alpha_S}{\pi} \gamma_c \to B_c(\alpha_S) \,, \tag{5.29}$$

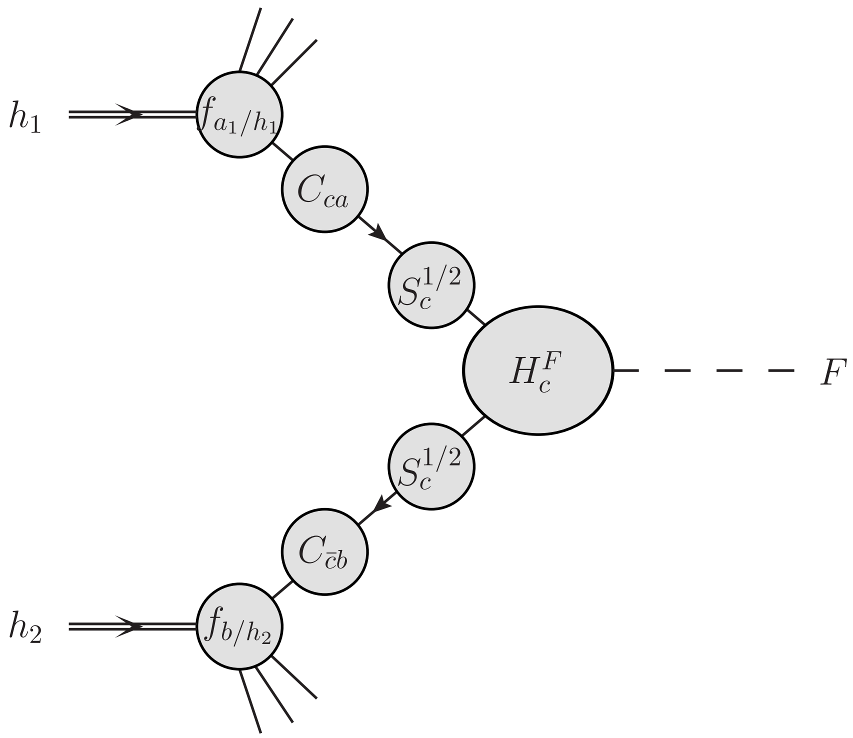

Figure 5.1: Pictorial representation of the resummation formula in Eq. (5.23) for DY-like processes. The function S_c is represented as collinear factors of the two initial-state partons, in order to point out that non-collinear soft-gluon emissions do not contribute to the cross section, in the low-q_T region.

where the functions $A_c(\alpha_S)$ and $B_c(\alpha_S)$ are perturbative series in α_S and play the same role as $A_c^{th}(\alpha_S)$ and $B_c^{th}(\alpha_S)$ for threshold resummation (see Chapter 3). The resulting all-order expression is given in Eq. (5.34).

We briefly review the content of the resummation formula. We recall that the Born-level factor $\sigma_{c\bar{c},F}$ in Eq. (5.23) is the lowest order cross section (scattering amplitude) for the partonic subprocess $c + \bar{c} \to F$, defined in Eq. (5.10). The remaining process dependence is embodied in the symbolic factor $\left[H^F C_1 C_2\right]$, with the following explicit form [93]:

$$\left[H^F C_1 C_2\right]_{c\bar{c};a_1a_2} = H_c^F(M, \boldsymbol{\Omega})\, C_{ca_1}(z_1; \alpha_S(b_0^2/b^2))\, C_{\bar{c}a_2}(z_2; \alpha_S(b_0^2/b^2))\,. \tag{5.30}$$

The hard-virtual function H_c^F receive contributions from the multi-loop radiative corrections to the elastic partonic-scattering process of Eq. (5.7). It includes, besides the virtual corrections, also the IR poles coming from the the real corrections and it is therefore finite. The functions $C_{ab}(z; \alpha_S)$ are universal and only depend on the parton flavours. We note that the scale of the coupling α_S is M^2 in the case of H_c^F and b_0^2/b^2 in the case of C_{ab}. This distinction, that can not be inferred from the simple NLO calculation reported in Section 5.1, is essential to define universal Sudakov form factors S_c and collinear functions C_{ab} and to capture all the process-dependence in the hard-virtual term H_c^F [93]. We finally note that the explicit expression of $\left[H^F C_1 C_2\right]$ in Eq. (5.30) is only valid in the case of processes that are initiated by $q\bar{q}$ annihilation, and it is more involved in the case of gluon-fusion. More details are given in Section 5.4. In Eq. (5.23) both the contribution from the $q\bar{q}$ annihilation channel ($c = q, \bar{q}$) and from the gluon fusion channel ($c = g$) are included, but one of these two channels may be absent ($\sigma^{(0)}_{c\bar{c}\to F} = 0$), depending on the final-state system F. For example, the production of a Higgs boson receives contribution only from the gluon-fusion channel, at the Born level, while the production of a lepton pair only from the $q\bar{q}$-annihilation channel.

The parton densities $f_{a/h}(b_0^2/b^2)$ in Eq. (5.23) are related to the parton densities $f_{a/h}(\mu_F^2)$ according to the DGLAP evolution equations (2.20). In Mellin space,

$$f_{a/h,N}(b_0^2/b^2) = \sum_b U_{ab,N}(b_0^2/b^2, \mu_F^2)\, f_{b/h,N}(\mu_F^2)\,, \tag{5.31}$$

where $U_{ab,N}$ is the QCD evolution operator, fulfilling the equations

$$\frac{dU_{ab,N}(\mu^2, \mu_0^2)}{d\ln\mu^2} = \sum_c \gamma_{ac,N}(\alpha_S(\mu^2))\, U_{cb,N}(\mu^2, \mu_0^2)\,. \tag{5.32}$$

This operator resums some large logarithms $\ln(b_0^2/b^2)$. In the simple case where there is a single species of partons, the solution of the evolution equation (5.32) is

$$U_N(b_0^2/b^2, \mu_F^2) = \exp\left\{-\int_{b_0^2/b^2}^{\mu_F^2} \frac{dq^2}{q^2}\gamma_N(\alpha_S(q^2))\right\}. \tag{5.33}$$

where γ_N is the anomalous dimension defined in Eq. (2.25).

The Sudakov form factor S_c is universal and only depends on the type of the colliding partons, two quarks ($c = q, \bar{q}$) or two gluons ($c = g$). Together with the PDFs evaluated at the scale $\mu = b_0/b$, it resums the large logarithms $\ln(M^2 b^2)$. It has the following all-order expression:

$$S_c(M, b) = \exp\left\{-\int_{b_0^2/b^2}^{M^2} \frac{dq^2}{q^2}\left[A_c(\alpha_S(q^2))\ln\frac{M^2}{q^2} + B_c(\alpha_S(q^2))\right]\right\}. \tag{5.34}$$

The resummation kernels $A_c(\alpha_S)$, $B_c(\alpha_S)$ in Eq. (5.34) and the hard functions $H_c^F(\alpha_S)$, $C_{ab}(z;\alpha_S)$ in Eq. (5.23) are perturbative series in α_S,

$$A_c(\alpha_S) = \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n A_c^{(n)} \,, \tag{5.35}$$

$$B_c(\alpha_S) = \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n B_c^{(n)} \,, \tag{5.36}$$

$$C_{ab}(z;\alpha_S) = \delta_{ab}(1-z) + \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n C_{ab}^{(n)}(z) \,, \tag{5.37}$$

$$H_c^F(\alpha_S) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n H_c^{F(n)} \,. \tag{5.38}$$

The process-independent Sudakov form factor in Eq. (5.34) is analogous to the radiative factor $\Delta_{a,N}$ in Eq. (3.28). The perturbative coefficients $A_c^{(1)}$, $A_c^{(2)}$, $B_c^{(1)}$ [99–101], $B_c^{(2)}$ [91,97] and $A_c^{(3)}$ [57] are explicitly known. The lowest order ones, $A_c^{(1)}$, $A_c^{(2)}$, $B_c^{(1)}$, are equal to the corresponding coefficients for threshold resummation, $A_c^{\mathrm{th}(1)}$, $A_c^{\mathrm{th}(2)}$, $B_c^{\mathrm{th}(1)}$, defined in Eqs. (3.24), (3.26), (3.45). These are the coefficients required at the NLL accuracy. The function S_c resums the logarithmic terms due to the flavour-conserving emission of real partons collinear to the initial-state hard partons, in the range $1/b^2 < q^2 < M^2$, from the soft up to the hard scale. Specifically, function $A_c(\alpha_S)$ accounts for soft-collinear emission and function $B_c(\alpha_S)$ for non-soft collinear emission.

The residual collinear emission at the soft scale scale $q^2 \sim 1/b^2$ is included in the partonic functions C_{ab}. In the 'hard' resummation scheme defined in [102] all the process dependence is collected in the H_c^F factor, while the C_{ab} functions are process-independent. Owing to flavour symmetry and charge-conjugation invariance of QCD, there are five independent quark functions C_{qa} $(a = g, q, q', \bar{q}, \bar{q}')$ and two independent gluon functions C_{ga} $(a = g, q)$. The lowest-order coefficients are explicitly known. The non-diagonal coefficients do not depend on the process and on the resummation scheme, while the diagonal ones are process-independent only in the hard scheme. In this scheme the first-order coefficients read:

$$C_{qq}^{(1)}(z) = \frac{1}{2}C_F(1-z)\,, \tag{5.39}$$

$$C_{gq}^{(1)}(z) = \frac{1}{2}C_F\, z\,, \tag{5.40}$$

$$C_{qg}^{(1)}(z) = \frac{1}{2}z(1-z)\,, \tag{5.41}$$

$$C_{gg}^{(1)}(z) = C_{q\bar{q}}^{(1)}(z) = C_{qq'}^{(1)}(z) = C_{q\bar{q}'}^{(1)}(z) = 0\,. \tag{5.42}$$

The coefficients in Eqs. (5.39)-(5.42) are equal to the $\mathcal{O}(\epsilon)$ part of the lowest-order Altarelli-Parisi splitting functions in Eqs. (5.13)-(5.16) and could therefore be reabsorbed in the PDFs, if a different factorisation scheme is employed. At the first order in α_{S}, at least for the $q\bar{q}$ channel, it would be enough to define the collinear counterterms in terms of the full ϵ-dependent Altarelli-Parisi probabilities, instead of the same functions at $\epsilon = 0$, as in the $\overline{\mathrm{MS}}$ scheme. Also the scale of the α_{S} coupling for $C_{ab}(z;\alpha_{\mathrm{S}})$ in Eq. (5.30) is the correct one ($\mu_F^2 = b_0^2/b^2$). Finally we note that the hard-virtual terms C^{DIS} in Eq. (3.12) and C^{DY} in Eq. (3.33) are analogous to the hard-virtual term H_c^F of Eq. (5.23). The analogue of the C_{ab} coefficients is instead absent in the case of threshold resummation, in which the enhanced non-logarithmic terms all appear at $z = 1$. The second-order coefficients are also known [103, 104]. Recently this result has been confirmed through an independent calculation based on SCET [105, 106].

In the next section we focus on the H_c^F coefficients and we give their explicit definition in terms of the QCD amplitudes that result from a standard fixed-order calculation. With a final-state consisting of colourless partons, the structure of the hard-virtual terms is rather simple. This allow us to fix the notation and to refer to it in Chapter 6, where the amplitudes are promoted to vectors in colour-space (see Chapter 4) and the hard-virtual factors to projections of colour-space operators on colour-states.

5.3 The hard-virtual term

All the information on the process-dependent virtual corrections is contained in the coefficient H_c^F, in the hard scheme. This term can be related, in a universal way, to the multiloop virtual amplitude $\mathcal{M}_{c\bar{c}\to F}$ of the elastic parton scattering of Eq. (5.7). In the center-of-mass frame of the initial-state partons, the system F has zero transverse momentum, in the absence of recoiling radiation. In the following we work with renormalised on-shell scattering amplitudes. Before performing the renormalisation, the QCD amplitude of the process in Eq. (5.7) contains UV and IR singularities, which can be regularised in the CDR scheme [84] in $d = 4 - 2\epsilon$ space-time dimensions. This introduces a perturbative dependence on powers of $\alpha_S^u \mu_0^{2\epsilon}$, where α_S^u is the bare coupling and μ_0 is the dimensional-regularisation scale. In the $\overline{\text{MS}}$ scheme, the renormalised scattering amplitude is obtained form the unrenormalised one by expressing the bare coupling in terms of the running coupling at the renormalisation scale μ_R^2, according to

$$\alpha_S^u \mu_0^{2\epsilon} S_\epsilon = \alpha_S(\mu_R^2)\, \mu_R^{2\epsilon} \left[1 - \alpha_S(\mu_R^2) \frac{\beta_0}{\epsilon} + \mathcal{O}\left(\alpha_S(\mu_R^2)^2\right) \right] . \tag{5.43}$$

The factor S_ϵ is

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon \gamma_E} . \tag{5.44}$$

The all-order perturbative expansion of the renormalised amplitude can be formally written as

$$\begin{aligned} \mathcal{M}_{c\bar{c}\to F}(p_1, p_2; \{q_i\}) = \left(\alpha_S(\mu_R^2)\mu^{2\epsilon}\right)^k \Big[&\mathcal{M}^{(0)}_{c\bar{c}\to F}(p_1, p_2; \{q_i\}) \\ &+ \sum_{n=1}^{\infty} \left(\frac{\alpha_S(\mu_R)}{2\pi} \right)^n \mathcal{M}^{(n)}_{c\bar{c}\to F}(p_1, p_2; \{q_i\}; \mu_R) \Big], \end{aligned} \tag{5.45}$$

where the value k of the overall power of α_S depends on the specific process. The leading-order factor $\alpha_S^k(\mu_R^2)$ is included in the normalisation of the amplitude, so that the rest of the expression is renormalisation group invariant. The momenta $\{q_i\}$ are the momenta of the n particles in the final-state system F, such that $(q_1 + q_2 + \ldots q_n)^2 = M^2$ and $\mathbf{q}_{1\perp} + \mathbf{q}_{2\perp} + \ldots \mathbf{q}_{n\perp} = 0$. The amplitude $\mathcal{M}^{(0)}_{a_1 a_2 \to F}$ is the Born-level contribution, $\mathcal{M}^{(n)}_{a_1 a_2 \to F}$ are the perturbative terms at the n-loop level, renormalised in the $\overline{\text{MS}}$ scheme. These terms are UV finite but still IR divergent in the $\epsilon \to 0$ limit.

The IR divergent terms have a universal structure, which is known at one-loop [66, 67, 85, 98], two-loops [67, 107, 108] and three-loop order for the class of processes in (5.7). It is then possible to define the auxiliary finite hard-virtual amplitude [102]

$$\widetilde{\mathcal{M}}_{c\bar{c}\to F}(p_1, p_2; \{q_i\}) = \left[1 - \widetilde{I}_c^{\text{DY}}(\epsilon, M) \right] \mathcal{M}_{c\bar{c}\to F}(p_1, p_2; \{q_i\}) , \tag{5.46}$$

where $\widetilde{I}_c$ is a process-independent IR subtraction factor. It only depends on the type c of the colliding partons and on the invariant mass M of the final-state system F, but not on its internal structure. The dependence on ϵ is fixed at the level of the poles, in such a way that the amplitude $\widetilde{\mathcal{M}}_{c\bar{c}\to F}$ is free of any pole in ϵ. The operator $[1-\widetilde{I}_c]$ thus remove every pole of $\mathcal{M}_{c\bar{c}\to F}$ and *some* of its IR finite terms. The factorisation formula in Eq. (5.46) can be regarded as the implicit definition of factor $\widetilde{I}_c$, order by order in $\alpha_{\rm S}$ and up to terms that are finite as ϵ vanishes. The all-order expansion of the IR subtraction factor is

$$\widetilde{I}_c^{\rm DY}(\epsilon, M) = \sum_{n=1}^{\infty} \left(\frac{\alpha_{\rm S}(\mu_R^2)}{2\pi}\right)^n \widetilde{I}_c^{{\rm DY}(n)}(\epsilon, M; \mu_R)\,. \tag{5.47}$$

The coefficients $\widetilde{I}_c^{{\rm DY}(n)}$ depend on the renormalisation scale through the dimensionless ratio M^2/μ_R^2. The amplitude $\widetilde{\mathcal{M}}_{c\bar{c}\to F}$ admits the same expansion of (5.45), in function of perturbative terms $\widetilde{\mathcal{M}}^{(n)}_{c\bar{c}\to F}$ that are free of ϵ-poles. Explicitly, the first two terms of the expansion are

$$\widetilde{\mathcal{M}}^{(0)}_{c\bar{c}\to F} = \mathcal{M}^{(0)}_{c\bar{c}\to F}\,, \tag{5.48}$$
$$\widetilde{\mathcal{M}}^{(1)}_{c\bar{c}\to F} = \mathcal{M}^{(1)}_{c\bar{c}\to F} - \widetilde{I}_c^{{\rm DY}(1)}(\epsilon, M; \mu_R)\,\mathcal{M}^{(0)}_{c\bar{c}\to F}\,. \tag{5.49}$$

At the Born-level, the two amplitudes are equal, being the LO amplitude free of IR divergencies. At higher orders in $\alpha_{\rm S}$, the subtracted amplitude $\widetilde{\mathcal{M}}_{c\bar{c}\to F}$ depends on the multiloop amplitudes and subtraction operators at equal or lower orders. The explicit expression of the first-order subtraction factor $\widetilde{I}_c^{(1)}$ is

$$\widetilde{I}_c^{{\rm DY}(1)}(\epsilon, M; \mu_R) = \widetilde{I}_c^{(1){\rm soft}}(\epsilon, M^2/\mu_R^2) + \widetilde{I}_c^{(1){\rm coll}}(\epsilon, M^2/\mu_R^2)\,, \tag{5.50}$$

with

$$\widetilde{I}_c^{(1){\rm soft}}(\epsilon, M^2/\mu_R^2) = -\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2} + i\pi\frac{1}{\epsilon}\right) C_a \left(\frac{\mu_R^2}{M^2}\right)^{\epsilon}, \tag{5.51}$$
$$\widetilde{I}_c^{(1){\rm coll}}(\epsilon, M^2/\mu_R^2) = -\frac{1}{\epsilon}\,\gamma_a \left(\frac{\mu_R^2}{M^2}\right)^{\epsilon}. \tag{5.52}$$

The $\widetilde{I}_c^{(1){\rm soft}}$ factor subtracts the soft-collinear double pole and the $\widetilde{I}_c^{(1){\rm coll}}$ factor subtracts the collinear (non-soft) single pole.

We can now give the definition of the hard-virtual resummation coefficient H_c^F in terms of the subtracted amplitude $\widetilde{\mathcal{M}}_{c\bar{c}\to F}$,

$$H_c^F(p_1, p_2; \alpha_{\rm S}(M^2)) = \frac{|\widetilde{\mathcal{M}}_{c\bar{c}\to F}(p_1, p_2; \{q_i\})|^2}{\alpha_{\rm S}^{2k}(M^2)\,|\widetilde{\mathcal{M}}^{(0)}_{c\bar{c}\to F}(p_1, p_2; \{q_i\})|^2}\,, \tag{5.53}$$

which is the last ingredient of the resummation formula in Eq. (5.23). We note that the IR subtraction operators in Eqs. (5.51),(5.52) are defined up to arbitrary terms that are finite in the limit $\epsilon \to 0$. A different choice would result in a different definition of the $H_c^F(\alpha_S)$ factor and of the $C_{ab}(z; \alpha_S)$ function. Any difference in the hard-collinear factor would be compensated by an opposite contribution to the constant factor of the partonic functions, i.e. the terms proportional to $\delta(1-z)$. The total result would be unchanged, but some process-dependent terms of virtual origin would be moved from the hard-virtual factor to the partonic functions. The definition in Eqs. (5.51),(5.52) is the only one possible in the hard-scheme, where all the process-dependent information is contained in the single factor $H_c^F(\alpha_S)$.

5.4 Gluon fusion

The resummation formula presented in Section 5.2 was originally proposed for processes where the production of the final-state system is controlled by quark-antiquark annihilation. The same formalism has been naively applied to the gluon fusion subprocess, under the assumption that spin correlations do not play any role if the observable is not sensitive to the azimuthal degree-of-freedom of the final-state system. While this is true for cross-sections evaluated at a fixed-order in perturbation theory, it turns out that the resummation formula for the gluon fusion channel has an additional structure [64] with respect to that of Eq. (5.23), due to the collinear evolution of the colliding hadrons into the gluon partonic initial states.

Without entering into the details, we just mention that the resummation formula in Eq. (5.23) is valid at all orders for the $q\bar{q}$ annihilation channel ($c = q$), with the symbolic factor $\left[H^F C_1 C_2\right]$ defined in Eq. (5.30). In the case of processes initiated by gluon fusion ($c = g$), the same factor has the following explicit form,

$$\begin{aligned}\left[H^F C_1 C_2\right]_{gg;a_1 a_2} &= H^F_{g;\mu_1\nu_1,\mu_2\nu_2}(p_1, p_2, \mathbf{\Omega}; \alpha_S(M^2)) \\ &\times C^{\mu_1\,\nu_1}_{g\,a_1}(z_1; p_1, p_2, \mathbf{b}; \alpha_S(b_0^2/b^2))\, C^{\mu_2\,\nu_2}_{g\,a_2}(z_2; p_1, p_2, \mathbf{b}; \alpha_S(b_0^2/b^2))\,, \quad (5.54)\end{aligned}$$

where the function H_g^F admits a perturbative expansion analogous to that in Eq. (5.38), with lowest-order normalisation

$$H_g^{F(0)\mu_1\nu_1,\mu_2\nu_2}\; g_{\mu_1\nu_1}\; g_{\mu_2\nu_2} = 1\,. \quad (5.55)$$

The analogue of Eq. (5.53) is

$$H_g^{F\ \mu_1\nu_1\mu_2\nu_2}(p_1,p_2,\mathbf{\Omega};\alpha_S(M^2)) = \\ d^{\mu_1}_{\mu_1'}\ d^{\nu_1}_{\nu_1'}\ d^{\mu_2}_{\mu_2'}\ d^{\nu_2}_{\nu_2'}\ \frac{\left[\widetilde{\mathcal{M}}^{\mu_1'\mu_2'}_{gg\to F}(p_1,p_2;\{q_i\})\right]^\dagger \widetilde{\mathcal{M}}^{\nu_1'\nu_2'}_{gg\to F}(p_1,p_2;\{q_i\})}{\alpha_S^{2k}(M^2)\ |\mathcal{M}^{(0)}_{gg\to F}(p_1,p_2;\{q_i\})|^2}\,, \quad (5.56)$$

where $d^{\mu\nu} = d^{\mu\nu}(p_1,p_2)$ is the polarization tensor in Eq. (5.59) and it projects onto the Lorentz indices in the transverse plane. The gluonic tensor $C^{\mu\nu}_{ga}$ in Eq.(5.54) is

$$C^{\mu\nu}_{g\,a}(z;p_1,p_2,\mathbf{b};\alpha_S) = d^{\mu\nu}(p_1,p_2)\ C_{g\,a}(z;\alpha_S) + D^{\mu\nu}(p_1,p_2;\mathbf{b})\ G_{g\,a}(z;\alpha_S)\,. \quad (5.57)$$

The first term in the r.h.s. of Eq. (5.57) is equivalent to the content of Eq.(5.30), while the second term is specific to gluon fusion and originates from the spin-dependent and q_T-dependent Altarelli-Parisi splitting probabilities $[\hat{P}_{ga}(z,\mathbf{q}_\perp)]^{\mu\nu}$ that control the collinear evolution of the PDFs for a polarised gluon. It is part of the collinear splitting function (tensor) of the gluon and it At the first perturbative order we can replace the splitting function $\hat{P}^{(1)}_{g\,a}(z)$, corresponding to an unpolarised gluon, with

$$\left[\hat{P}^{(1)}_{ga}(z,\mathbf{q}_\perp)\right]^{\mu\nu} = -g^{\mu\nu}\ \hat{P}^{(1)}_{ga}(z) + \left(g^{\mu\nu} + 2\,\frac{q_T^\mu q_T^\nu}{\mathbf{q}_\perp^2}\right) 2\,G^{(1)}_{ga}(z)\,, \quad (5.58)$$

where the coefficients $G^{(1)}_{g\,a}(z)$, given in Eq. (5.61), control gluon spin correlations that have no analogue in quark-antiquark annihilation processes. The Fourier transform of Eq.(5.58) to $\mathbf{b}$-space produces the $C^{\mu\nu}_{ga}$ tensors of Eq. (5.57). If $b^\mu = (0,\mathbf{b},0)$ is the impact-parameter vector in the four-dimensional notation ($b^\mu b_\mu = -\mathbf{b}^2$), then the two tensors in Eq. (5.57) read

$$d^{\mu\nu}(p_1,p_2) = -g^{\mu\nu} + \frac{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu}{p_1\cdot p_2}\,, \quad (5.59)$$

$$D^{\mu\nu}(p_1,p_2;\mathbf{b}) = d^{\mu\nu}(p_1,p_2) - 2\,\frac{b^\mu\, b^\nu}{\mathbf{b}^2}\,. \quad (5.60)$$

The gluonic coefficient functions $C_{g\,a}(z;\alpha_S)$ are given in Eq. (5.37): they start at $\mathcal{O}(\alpha_S^0)$ and have first-order coefficients in the hard scheme as defined in Eqs. (5.39)-(5.42). The perturbative expansion of the coefficient functions G_{ga} instead starts at $\mathcal{O}(\alpha_S)$,

$$G_{g\,a}(z;\alpha_S) = \frac{\alpha_S}{\pi}\,G^{(1)}_{ga}(z) + \sum_{n=2}^{\infty}\left(\frac{\alpha_S}{\pi}\right)^n G^{(n)}_{ga}(z)\,, \quad (5.61)$$

with first-order coefficients that are resummation-scheme independent and equal to [64]

$$G^{(1)}_{ga}(z) = C_a \, \frac{1-z}{z} \, , \tag{5.62}$$

where C_a is the Casimir coefficient of parton a.

It can be shown that, for a final-state system of spin 0 (e.g. $F = H$), the resummation factor is [64]

$$\begin{aligned}
\left[H^F C_1 C_2\right]_{gg;a_1 a_2} &= g_{\mu_1\nu_1} g_{\mu_2\nu_2} \; H_g^{F\;\mu_1\nu_1\mu_2\nu_2}(\alpha_S(M^2)) \\
&\times \left[\, C_{ga_1}(z_1;\alpha_S(b_0^2/b^2))\, C_{ga_2}(z_2;\alpha_S(b_0^2/b^2)) \right. \\
&+ \left. \; G_{ga_1}(z_1;\alpha_S(b_0^2/b^2))\, G_{ga_2}(z_2;\alpha_S(b_0^2/b^2)) \,\right] .
\end{aligned} \tag{5.63}$$

We note that the term $G_{ga}(\alpha_S) G_{gb}(\alpha_S)$ in Eq. (5.63) starts at order $\mathcal{O}(\alpha_S^2)$ and it therefore affects the computation of the QCD radiative corrections starting at the NNLO+NNLL accuracy.

Chapter 6

Heavy-quark pair production

We consider the inclusive production of a pair of heavy quarks (Q) in hadron-hadron collisions. The bulk of the cross section is produced in the kinematical region where the transverse momentum q_{T} of the pair is smaller than the mass m of the heavy quark. In the following we are interested in the small-q_{T} region, namely, the region where $q_{\mathrm{T}} \ll m$ (including the limit $q_{\mathrm{T}} \to 0$). From the phenomenological point of view, the most relevant process is the production of a pair of top-antitop quarks, because of its topical importance in the context of both Standard Model (SM) and beyond-SM physics. The theoretical efforts for obtaining precise predictions for $t\bar{t}$ production at hadron colliders started almost three decades ago with the calculation of the NLO QCD corrections to this process [109–111]. Recently the calculation of the next-to-next-to-leading order (NNLO) QCD corrections to the inclusive $t\bar{t}$ cross section was completed [112–115], and soft gluon effects have been included [116]. Besides the total cross section, the differential distributions are of a great importance for precision studies, and the q_{T} spectrum of the $t\bar{t}$ pair is one important example. First measurements of the $t\bar{t}$ transverse-momentum distribution based on the $\sqrt{s} = 7$ data sample at the LHC have recently been presented [117, 118].

The all-order resummation for the heavy-quark process has been discussed only very recently by H. X. Zhu et al. in Refs. [62, 63]. The analysis of Refs. [62, 63] is limited to the study of the q_{T} cross section after integration over the azimuthal angles of the produced heavy quarks. In this chapter we summarise the results of our independent study of transverse-momentum resummation for $Q\bar{Q}$ production [119]. We present our all-order resummation formalism for $Q\bar{Q}$ production, and we explicitly perform resummation up to the next-to-next-to-leading lofarithmic (NNLL) level, by including the explicit control of *all* the contributions up to the next-to-leading order (NLO) in the perturbative expansion. Our formalism and results are valid at the fully-differential level with respect to the kinematics of the produced heavy quarks. In particular, we consider the explicit dependence on the azimuthal angles of the heavy quarks and we have full control, at the resummed level, of the ensuing *azimuthal correlations* in the small-q_T region.

In the previous chapter we have presented the (process-independent) structure of the transverse-momentum resummation for a general class of hard-scattering processes in which the produced high-mass system F in the final state is formed by a set of colourless particles. The $Q\bar{Q}$ production process definitely belongs to a different class of processes, since the produced final-state heavy quarks carry colour charge and, therefore, they act as additional source of QCD radiation. The q_T of the $Q\bar{Q}$ pair depends on initial-state radiation, on final-state radiation and on quantum (and colour flow) interferences between radiation from the initial and final states. These physical differences between $Q\bar{Q}$ production and the production of a colourless system F lead to very relevant technical and conceptual complications in the context of transverse-momentum resummation for $Q\bar{Q}$ production. An important issue regards the presence of possible contributions from factorization-breaking effects of collinear radiation [120–123]. Other complications, which already arise in the context of threshold resummation for the $Q\bar{Q}$ total cross section [40, 53, 124–126], regard the effect of non-abelian colour correlations produced by initial-state and final-state interferences. Additional important complications and effects, which are specific of transverse-momentum resummation, regard the azimuthal-angle distribution of the $Q\bar{Q}$ pair. In the case of the DY process, q_T resummation has no effect on the azimuthal correlation between the produced leptons, since the q_T broadening of the lepton pair is entirely due to QCD radiation from the initial-state ($q\bar{q}$) partons. In contrast, the q_T of the $Q\bar{Q}$ pair is also due to radiation from Q and $\bar{Q}$, and this leads to q_T-dependent azimuthal correlations.

6.1 The $Q\bar{Q}$ transverse-momentum cross section at NLO

We consider the production of two heavy quarks Q, $\bar{Q}$ of measured momenta p_3^μ, p_4^μ and invariant mass $m_3^2 = p_3^2$, $m_4^2 = p_4^2$, from the collisions of two hadrons h_1, h_2 of momenta P_1^μ, P_2^μ,

$$h_1(P_1) + h_2(P_2) \to Q(p_3) + \bar{Q}(p_4) + X\,, \tag{6.1}$$

with the associated production of QCD radiation X. If we consider the $Q\bar{Q}$ pair, the total momentum $q^\mu = (p_3^\mu + p_4^\mu)$ is measured and with it the invariant mass $q^2 = M^2$, the transverse-momentum $\mathbf{q}_\perp$ and the rapidity y in the centre-of-mass frame of the hadronic collision. In addition, the momenta p_3^μ and p_4^μ determine two angular variables $\mathbf{\Omega}$ of the heavy-quark pair. With this notation the two sets of variables $(P_1^\mu, P_2^\mu, p_3^\mu, p_4^\mu)$ and $(\mathbf{q}_\perp, y, M, \mathbf{\Omega})$ are equivalent and Eq. (6.1) can be written as

$$h_1(P_1) + h_2(P_2) \to Q\bar{Q}(\mathbf{q}_\perp, y, M, \mathbf{\Omega}) + X\,. \tag{6.2}$$

We are using the same notation of Section 5.1 (cfr. Eq. (6.2) with Eq. (5.1)), with the composite final-state *colourless* system F replaced by the heavy-quark pair, so that we can easily refer to the results of the previous Chapter and reuse them with the formal replacement $F \to Q\bar{Q}$. We use the same kinematical variables, with $\hat{s}$ and $\hat{y}$ defined in Eq. (5.4) and the scaling variables x_1, x_2 and z_1, z_2 defined in Eqs. (5.2) and (5.2), respectively. The Born-level momenta $(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu)$, with $p_1^\mu = x_1 P_1^\mu$ and $p_2^\mu = x_2 P_2^\mu$, are completely determined, in the partonic center-of-mass frame, by the set of variables $(M, \mathbf{\Omega})$ and the same is true for the Lorentz scalars $p_i \cdot p_j$, with $i, j = 1, \ldots 4$. Finally $p_3^2 = m_3^2$ and $p_4^2 = m_4^2$ (with $m_3^2 \sim m_4^2$) are the invariant masses of the heavy quarks.

We want to calculate the fully-differential cross section

$$\begin{aligned}\frac{d\sigma_{h_1 h_2 \to Q\bar{Q}+X}(s; \mathbf{q}_\perp, M, y, \mathbf{\Omega})}{d^2\mathbf{q}_\perp\, dM^2\, dy\, d\mathbf{\Omega}} &= \frac{1}{s} \sum_{a_1, a_2} \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} \\ \times f_{a_1/h_1}(x_1/z_1, \mu_F^2)\, f_{a_2/h_2}(x_2/z_2, \mu_F^2)\, &\frac{d\hat{\sigma}_{a_1 a_2 \to Q\bar{Q}+X}(\mathbf{q}_\perp, z_1, z_2; M, \mathbf{\Omega}; \mu_F)}{d^2\mathbf{q}_\perp\, dz_1\, dz_2\, d\mathbf{\Omega}}\,,\end{aligned} \tag{6.3}$$

in the kinematical region $q_{\mathrm{T}} \ll m_3^2, m_4^2$. To this purpose, we need to calculate the logarithmically enhanced contribution to the partonic cross section $d\hat{\sigma}_{a_1 a_2 \to Q\bar{Q}+X}$ and to possibly resum them to all orders. The QCD radiative corrections to the cross section of Eq. (5.6), due to the initial-state partons a_1, a_2, contribute to Eq. (6.3) as well, but in addition we must include the QCD radiation from the $Q\bar{Q}$ pair. If we calculate the NLO enhanced contributions in the vanishing-q_{T} limit we find a structure similar to that of Eq. (5.9), where the probability $W^{\mathrm{DY}(1)}_{c\bar{c};a_1 a_2}$ is replaced by a more general single-emission probability $W^{(1)}_{c\bar{c};a_1a_2;Q\bar{Q}}$. The heavy quarks are massive QCD emitters and therefore the additional radiation does not produce poles nor singular terms in the collinear limit. The hard-collinear singular terms at $z_1 < 1$ or $z_2 < 1$, i.e. the second and third lines of Eq. (5.11), are not affected by the replacement $F \to Q\bar{Q}$ and the additional contributions comes from the near-threshold region ($z_1 = z_2 = 1$). In other terms the single-emission probability reads:

$$W^{(1)}_{c\bar{c};a_1a_2}(\mathbf{q}_\perp, z_1, z_2; M, \mathbf{\Omega}; \mu_F) = W^{\mathrm{DY}(1)}_{c\bar{c};a_1a_2}(q_{\mathrm{T}}^2, z_1, z_2; M; \mu_F) \\ + W^{(1)}_{\mathrm{HQ}}(\mathbf{q}_\perp; M, \mathbf{\Omega})\, \delta(1-z_1)\, \delta(1-z_2)\, \delta_{ca_1}\delta_{\bar{c}a_2}\,. \quad (6.4)$$

It follows that the same prescriptions discussed in Chapter 5 to organise the hard-collinear singular terms are still valid: the natural choice for the factorisation scale is $\mu_F^2 = b_0^2/b^2$ while the non-soft non-logarithmic terms are controlled by the same universal collinear functions $C_{ab}(z;\alpha_{\mathrm{S}})$ found for DY-like production.

The function $W^{(1)}_{\mathrm{HQ}}(\mathbf{q}_\perp)$ in Eq. (6.4) includes the effects of large-angle soft-gluon exchanges between the four hard partons. Non-collinear terms appear starting with two hard partons in the matrix elements and they entail colour-correlation effects due to quantum-interference, at the level of the squared amplitude. When up to three partons are present at the Born level, these effects cancel out due to colour conservation (at least in the singular contribution) so that the resulting colour structure simplifies into the flavour coefficient C_c and γ_c of the hard partons. This is the case of the processes treated in Chapters 3 and 5: the fixed-order results and the resummation formulae fulfil a scalar factorisation. When four partons are involved, the colour flow is more complicated and the colour algebra does not factorise into diagonal terms, so that the final result is a sum over the colour states of the LO amplitude. In Chapter 4 we have presented our NLO result for the one-particle inclusive cross section in a way that allows to keep colour correlations under control, by working in the abstract colour space defined in Chapter 2. Along the same lines, we define the four colour charges $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$ associated to the partons c (and a_1), $\bar{c}$ (and a_2), Q, $\bar{Q}$ respectively. We promote the multiloop amplitude $\mathcal{M}_{c\bar{c}\to F}$ and the hard-virtual amplitude $\widetilde{\mathcal{M}}_{c\bar{c}\to F}$ of Section 5.3 to colour-space amplitudes $|\mathcal{M}_{c\bar{c}\to Q\bar{Q}}\rangle$ and $|\widetilde{\mathcal{M}}_{c\bar{c}\to Q\bar{Q}}\rangle$, with perturbative expansions defined as in Eq. (5.45), and the real-emission probability $W_{c\bar{c};a_1a_2}(\alpha_{\mathrm{S}})$ to the colour-space operator

$$\mathbf{W}_{c\bar{c};a_1a_2}(\alpha_{\mathrm{S}}) = \sum_{n=1}^{\infty} \left(\frac{\alpha_{\mathrm{S}}}{\pi}\right)^n \mathbf{W}^{(n)}_{c\bar{c};a_1a_2}\,. \quad (6.5)$$

The $\mathcal{O}(\alpha_S)$ radiative corrections are treated as in Section (4.1), with QCD matrix elements for single emission replaced by the eikonal factor in colour space projected on Born-level matrix elements. The amplitude is then squared and the result can be organised as the sum of three terms. One term is associated to parton a_1, one term to parton a_2 and one term to the large-angle colour flow, according to the eikonal factor squared:

$$- \sum_{i,j=1}^{4} \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)} \mathbf{T}_i \cdot \mathbf{T}_j = \frac{2\, p_1 \cdot p_2}{(p_1+p_2)\cdot k} \left(\frac{\mathbf{T}_1^2}{p_1 \cdot k} + \frac{\mathbf{T}_2^2}{p_2 \cdot k} \right) - |\boldsymbol{J}_{\mathrm{HQ}}(k)|^2 \,, \tag{6.6}$$

where k^μ is the momentum of the soft-gluon. The first two terms account for radiation that is soft and collinear to the two initial-state partons and they are diagonal in colour space (proportional to the Casimir $\mathbf{T}_1^2 = \mathbf{T}_2^2 = C_c$). If we improve their eikonal spectrum with the full ϵ-dependent Altarelli-Parisi splitting functions of Eq. (5.12) and we integrate over the available phase-space, we get exactly the radiative corrections presented in Section 5.1. The third term describes large-angle soft-gluon exchanges between the four hard partons and therefore it is not diagonal in colour space. It is proportional to the projection of the colour-space operator

$$|\boldsymbol{J}_{\mathrm{HQ}}(k)|^2 = \sum_{j=3,4} \frac{m_j^2}{(p_j \cdot k)^2} \mathbf{T}_j^2 + \frac{2\, p_3 \cdot p_4}{p_3 \cdot k \; p_4 \cdot k} \mathbf{T}_3 \cdot \mathbf{T}_4 + \sum_{\substack{i=1,2\\ j=3,4}} \frac{2}{p_i \cdot k} \left(\frac{p_i \cdot p_j}{p_j \cdot k} - \frac{p_1 \cdot p_2}{(p_1+p_2)\cdot k} \right) \mathbf{T}_i \cdot \mathbf{T}_j \tag{6.7}$$

on the Born-level amplitude $|\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}\rangle$. The expression in Eq. (6.7) satisfies two important consistency requirements. First of all it vanishes if the colour charges of the heavy quarks are switched off, i.e. if $\mathbf{T}_3 = \mathbf{T}_4 = 0$: in this case the final-state system is colourless and the scattering belongs to the class of processes treated in the previous chapter. Secondly, each colour-interference term is finite in the collinear limits $k^\mu \sim \lambda p_i^\mu$ and it diverges only in the soft limit $k \to 0$, in agreement with the physical argument that heavy quarks, being massive, do not contribute with additional singular terms of collinear origin. This also implies that the phase-space integral of the third term produces at most single-logarithmic terms, associated to additional IR (soft) $1/\epsilon$ poles in the real and virtual corrections (cfr. Eqs. (6.13) and Eqs.(6.12)).

We can write the singular $\mathcal{O}(\alpha_S)$ radiative corrections in a form similar to that of Eq. (4.17): the single-emission probability $W^{(1)}_{c\bar{c};a_1a_2}$ of Eq. (6.4) is equal to the projection of $\mathbf{W}^{(1)}_{c\bar{c};a_1a_2}$ on the Born-level colour amplitude $|\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}\rangle$ and the hard-virtual term $H_c^{F(1)}$ of Eq. (5.9) is replaced by the IR-finite interference term $\Big(\langle\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}|\widetilde{\mathcal{M}}^{(1)}_{c\bar{c}\to Q\bar{Q}}\rangle + c.c.\Big)$. The singular partonic cross section at NLO in $\mathbf{q}_\perp$-space reads :

$$\frac{d\hat{\sigma}^{\rm sing}_{a_1a_2\to Q\bar{Q}+X}}{d^2\mathbf{q}_\perp\, dz_1\, dz_2\, d\mathbf{\Omega}} = \frac{M^2}{\pi}\sum_{c=q,\bar{q},g}[d\sigma^{(0)}_{c\bar{c},Q\bar{Q}}]\Big\{\delta(q_{\rm T}^2)\,\delta(1-z_1)\,\delta(1-z_2)\,\delta_{ca_1}\delta_{\bar{c}a_2}$$
$$+\frac{\alpha_{\rm S}}{2\pi}\,\frac{\langle\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}|\widetilde{\mathcal{M}}^{(1)}_{c\bar{c}\to Q\bar{Q}}\rangle + c.c.}{|\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}|^2}\,\delta(q_{\rm T}^2)\,\delta(1-z_1)\,\delta(1-z_2)\,\delta_{ca_1}\delta_{\bar{c}a_2}$$
$$+\frac{\alpha_{\rm S}}{\pi}\,\frac{\langle\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}|\mathbf{W}^{(1)}_{c\bar{c};a_1a_2}(\mathbf{q}_\perp,z_1,z_2;M,\mathbf{\Omega};\mu_F)|\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}\rangle}{|\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}|^2} + \mathcal{O}(\alpha_{\rm S}^2)\Big\}\,. \quad (6.8)$$

The $\mathcal{O}(\alpha_{\rm S})$ coefficient of the hard-virtual amplitude in the second line of Eq. (6.8) is

$$|\widetilde{\mathcal{M}}^{(1)}_{c\bar{c}\to Q\bar{Q}}\rangle = |\mathcal{M}^{(1)}_{c\bar{c}\to Q\bar{Q}}\rangle - \widetilde{\mathbf{I}}^{(1)}_c(\epsilon,M,\mathbf{\Omega};\mu_R)\,|\mathcal{M}^{(0)}_{c\bar{c}\to Q\bar{Q}}\rangle\,, \quad (6.9)$$

with subtraction operator

$$\widetilde{\mathbf{I}}^{(1)}_c(\epsilon,M,\mathbf{\Omega};\mu_R) = \widetilde{I}^{\rm DY(1)}_c(\epsilon,M;\mu_R) + \widetilde{\mathbf{I}}^{(1)}_{\rm HQ}(\epsilon,M,\mathbf{\Omega};\mu_R)\,, \quad (6.10)$$

and the single-emission probability in the third line of Eq. (6.8) is

$$\mathbf{W}^{(1)}_{c\bar{c};a_1a_2}(\mathbf{q}_\perp,z_1,z_2;M,\mathbf{\Omega};\mu_F) = W^{\rm DY(1)}_{c\bar{c};a_1a_2}(q_{\rm T}^2,z_1,z_2;M;\mu_F)$$
$$+\,\mathbf{W}^{(1)}_{\rm HQ}(\mathbf{q}_\perp;M,\mathbf{\Omega})\,\delta(1-z_1)\,\delta(1-z_2)\,\delta_{ca_1}\delta_{\bar{c}a_2}\,. \quad (6.11)$$

The infrared corrections due to DY-like collinear emission from the initial-state partons a_1,a_2 are included in the subtraction operator $\widetilde{I}^{\rm DY}_c(\epsilon)$ and in the probability $W^{\rm DY(1)}_{c\bar{c};a_1a_2}(q_{\rm T}^2,z_1,z_2)$, defined in Chapter (5). The colour-space operators $\widetilde{\mathbf{I}}^{(1)}_{\rm HQ}(\epsilon)$ in Eqs. (6.9), (6.10) and $\mathbf{W}^{(1)}_{\rm HQ}(\mathbf{q}_\perp)$ in Eq. (6.11), that account for the additional large-angle soft-gluon emissions due to the heavy quarks, are equal to

$$\widetilde{\mathbf{I}}^{(1)}_{\rm HQ}(\epsilon,M,\mathbf{\Omega};\mu_R^2) = -\frac{1}{2}\left(\frac{M^2}{\mu_R^2}\right)^{-\epsilon}\left\{-\frac{4}{\epsilon}\,\mathbf{\Gamma}^{(1)}_{\rm T}(M,\mathbf{\Omega}) + \mathbf{F}^{(1)}(M,\mathbf{\Omega})\right\}\,, \quad (6.12)$$

$$\mathbf{W}^{(1)}_{\rm HQ}(\mathbf{q}_\perp;M,\mathbf{\Omega}) = \left(\frac{1}{q_{\rm T}^2}\right)_+\Big[\mathbf{\Gamma}^{(1)}_{\rm T}(M,\mathbf{\Omega}) + \mathbf{\Gamma}^{(1)\,\dagger}_{\rm T}(M,\mathbf{\Omega})\Big]$$
$$+\frac{1}{q_{\rm T}^2}\Big[\mathbf{R}^{(1)}(\hat{\mathbf{q}}_\perp;M,\mathbf{\Omega}) - \langle\mathbf{R}^{(1)}\rangle(M,\mathbf{\Omega})\Big]\,, \quad (6.13)$$

where the three operators $\boldsymbol{\Gamma}_{\rm T}^{(1)}$, $\mathbf{F}^{(1)}$, $\mathbf{R}^{(1)}$, whose explicit expressions are given in Section 6.6, are determined by the phase-space integral of the expression in Eq. (6.7) and therefore vanish if $\mathbf{T}_3 = \mathbf{T}_4 = 0$. They are all defined at Born-level kinematics $(M, \boldsymbol{\Omega}) \leftrightarrow (p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu)$. The operator $\mathbf{R}^{(1)}$ additionally depends on the azimuthal degree-of-freedom $\hat{\mathbf{q}}_\perp$ of the transverse momentum $\mathbf{q}_\perp$ and the operator $\langle \mathbf{R}^{(1)} \rangle$, defined in Eq. (6.16) in terms of $\boldsymbol{\Gamma}_{\rm T}^{(1)}$ and $\mathbf{F}^{(1)}$, is its azimuthal average over $\hat{\mathbf{q}}_\perp$. It follows that the difference $[\mathbf{R}^{(1)}(\hat{\mathbf{q}}_\perp) - \langle \mathbf{R}^{(1)} \rangle]$ is zero at $\mathbf{q}_\perp = 0$ and the second line in the r.h.s. of Eq. (6.13) is integrable in $q_{\rm T}$ in the same limit.

The operator $\boldsymbol{\Gamma}_{\rm T}^{(1)}$, defined in Eq. (6.36), controls the soft poles of the subtraction operator $\widetilde{\mathbf{I}}_{\rm HQ}^{(1)}$ and the ensuing single-logarithmic term (the plus-distribution of $1/q_{\rm T}^2$) of the operator $\mathbf{W}_{\rm HQ}^{(1)}$. The $1/\epsilon$ poles determined by the expression in Eq. (6.36) are in agreement with the one-loop singular behaviour of QCD amplitudes with massive partons [127, 128]: the structure of the real poles is equal in magnitude and opposite in sign to the structure of virtual poles, so that the hard-virtual amplitude $|\widetilde{\mathcal{M}}_{c\bar{c}\to Q\bar{Q}}^{(1)}\rangle$ in Eq. (6.9) is explicitly finite.

The operators $\mathbf{F}^{(1)}$ and $\mathbf{R}^{(1)}$, defined in Eqs. (6.38) and (6.43) respectively, both control finite non-logarithmic terms. The integral of Eq. (6.7) gives $\mathbf{R}^{(1)}(\hat{\mathbf{q}}_\perp)/q_{\rm T}^2$, that we can decompose as

$$\frac{1}{(q_{\rm T}^2)^{1+\epsilon}}\, \mathbf{R}^{(1)}(\hat{\mathbf{q}}_\perp; \epsilon) = \frac{1}{(q_{\rm T}^2)^{1+\epsilon}}\, \langle \mathbf{R}^{(1)}(\epsilon) \rangle + \frac{1}{q_{\rm T}^2} \left[\mathbf{R}^{(1)}(\hat{\mathbf{q}}_\perp) - \langle \mathbf{R}^{(1)} \rangle \right]. \quad (6.14)$$

The second term in the r.h.s. of Eq. (6.14) is finite, it can be calculated at $\epsilon = 0$ and it directly corresponds to the $1/q_{\rm T}^2$ term of $\mathbf{W}_{\rm HQ}^{(1)}(\mathbf{q}_\perp)$, in Eq. (6.13). The first term in the r.h.s. of Eq. (6.14) has a pole at $q_{\rm T} = 0$ and must be regularised in $d = 4 - 2\epsilon$ dimensions. From the identities

$$\frac{1}{(q_{\rm T}^2)^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(q_{\rm T}^2) + \left(\frac{1}{q_{\rm T}^2}\right)_+ + \mathcal{O}(\epsilon)\,, \quad (6.15)$$

$$\langle \mathbf{R}^{(1)}(\epsilon) \rangle = \left[\boldsymbol{\Gamma}_{\rm T}^{(1)} + \boldsymbol{\Gamma}_{\rm T}^{(1)\,\dagger} \right] - \frac{\epsilon}{4}\left[\mathbf{F}^{(1)} + \mathbf{F}^{(1)\,\dagger} \right] + \mathcal{O}(\epsilon^2)\,, \quad (6.16)$$

we obtain the $[1/q_{\rm T}^2]_+$ term of $\mathbf{W}_{\rm HQ}^{(1)}(\mathbf{q}_\perp)$ in Eq. (6.13) and $\widetilde{\mathbf{I}}_{\rm HQ}^{(1)}$ in Eq. (6.12). The operator $\mathbf{F}^{(1)}$ is thus the coefficient of the finite $\delta(q_{\rm T}^2)$ terms coming from Eq. (6.14) and it naturally belongs to the hard-virtual amplitude. In defining the hard-virtual amplitude, we subtract the soft-virtual amplitude from the (total) virtual amplitude and in doing so we rely on unitarity: the virtual corrections in the soft limit are equal in magnitude and opposite in sign to the $\delta(q_{\mathrm{T}}^2)$ contribution of the real corrections. The second term in Eq. (6.14) is transformed to a constant in $\mathbf{b}$-space, but it does not belong to the $\delta(q_{\mathrm{T}}^2)$ terms and therefore it must be included in the real-emission probability of Eq. (6.13), together with the $[1/q_{\mathrm{T}}^2]_+$ term.

The operators $\widetilde{\mathbf{I}}^{(1)}_{\mathrm{HQ}}$ and $\mathbf{W}^{(1)}_{\mathrm{HQ}}(\mathbf{q}_\perp)$ provide all the information that we need in order to extend the formula in Eq. (5.23), to include the production of a $Q\bar{Q}$ pair. First we Fourier-transform the partonic cross section in Eq. (6.8) to $\mathbf{b}$-space, to get a factorised result, ready to be resummed to all orders. The first two lines of Eq. (6.8) are constant in the conjugate space and the third line requires the transform of Eq. (6.11), i.e. of $W^{\mathrm{DY}(1)}_{c\bar{c};a_1a_2}(q_{\mathrm{T}}^2, z_1, z_2)$ and of $W^{\mathrm{HQ}(1)}_{c\bar{c};a_1a_2}(\mathbf{q}_\perp)$. The former leads to the results of the previous chapter, with the definition of the S_c and C_{ab} functions to describe collinear soft and hard radiation from the initial-state partons and the definition of a running factorisation scale $\mu_F^2 = b_0^2/b^2$. As regards $W^{\mathrm{HQ}(1)}_{c\bar{c};a_1a_2}(\mathbf{q}_\perp)$, we can transform the $[1/q_{\mathrm{T}}^2]_+$ term in Eq. (6.13) with the approximation in Eq. (5.20) to get

$$-\int_{b_0^2/b^2}^{M^2} \frac{dk_{\mathrm{T}}^2}{k_{\mathrm{T}}^2} \left[\mathbf{\Gamma}^{(1)}_{\mathrm{T}}(M, \mathbf{\Omega}) + \mathbf{\Gamma}^{(1)\,\dagger}_{\mathrm{T}}(M, \mathbf{\Omega}) \right] \tag{6.17}$$

while the $1/q_{\mathrm{T}}^2$ term in Eq. (6.13) must be transformed exactly and gives

$$\int \frac{d^2\mathbf{b}}{(2\pi)^2}\, e^{i\mathbf{b}\cdot\mathbf{q}_\perp}\, \mathbf{D}^{(1)}(\hat{\mathbf{b}}; M, \mathbf{\Omega}) + \mathcal{O}(\mathbf{q}_\perp)\,. \tag{6.18}$$

The explicit expression of the operator $\mathbf{D}^{(1)}$ is given in Eq. (6.46). It depends on the azimuthal degree-of-freedom $\hat{\mathbf{b}}$ of the impact-parameter $\mathbf{b}$ and has zero azimuthal average (with respect to $\hat{\mathbf{b}}$).

In order to include the contributions of Eqs. (6.12), (6.17), (6.18) in the $\mathbf{b}$-space formula of the singular cross section we proceed on the same lines of Chapter 4: we replace the hard-virtual term H_c^F of Eq. (5.23) with a factor $(\mathbf{H}\,\mathbf{\Delta})_c$ acting in colour space, that entails colour interferences between a soft-real radiative factor $\mathbf{\Delta}$ and the hard-virtual amplitude $|\widetilde{\mathcal{M}}_{q\bar{q}\to Q\bar{Q}}\rangle$. Explicitly:

$$(\mathbf{H}\,\mathbf{\Delta})_q = \frac{\langle \widetilde{\mathcal{M}}_{q\bar{q}\to Q\bar{Q}} |\, \mathbf{\Delta}(\mathbf{b}; M, \mathbf{\Omega})\, | \widetilde{\mathcal{M}}_{q\bar{q}\to Q\bar{Q}} \rangle}{\alpha_{\mathrm{S}}^2(M^2)\, |\mathcal{M}^{(0)}_{q\bar{q}\to Q\bar{Q}}|^2}\,, \tag{6.19}$$

for processes initiated at the Born-level by $q\bar{q}$ annihilation and

$$(\mathbf{H}\,\mathbf{\Delta})_g^{\mu_1\nu_1,\mu_2\nu_2} = d^{\mu_1}_{\mu_1'}\, d^{\nu_1}_{\nu_1'}\, d^{\mu_2}_{\mu_2'}\, d^{\nu_2}_{\nu_2'}\, \frac{\langle \widetilde{\mathcal{M}}^{\mu_1'\mu_2'}_{gg\to Q\bar{Q}} |\, \mathbf{\Delta}(\mathbf{b}; M, \mathbf{\Omega})\, | \widetilde{\mathcal{M}}^{\nu_1'\nu_2'}_{gg\to Q\bar{Q}} \rangle}{\alpha_{\mathrm{S}}^2(M^2)\, |\mathcal{M}^{(0)}_{gg\to Q\bar{Q}}|^2}\,, \tag{6.20}$$

for processes initiated by gluon fusion. The $\widetilde{\mathbf{I}}^{(1)}_{\rm HQ}$ operator of Eq. (6.12) enters into the definition of the NLO hard-virtual amplitude, according to Eqs.(6.9) and (6.10), whereas the expressions in Eqs. (6.17) and (6.18) define the radiative factor $\mathbf{\Delta}$ at order $\mathcal{O}(\alpha_{\rm S})$:

$$\begin{aligned}\mathbf{\Delta}^{(\rm f.o.)}(\mathbf{b}; M, \mathbf{\Omega}) = 1 - \frac{\alpha_{\rm S}}{\pi} \int_{b_0^2/b^2}^{M^2} \frac{dk_{\rm T}^2}{k_{\rm T}^2} \left[\mathbf{\Gamma}_{\rm T}^{(1)}(M, \mathbf{\Omega}) + \mathbf{\Gamma}_{\rm T}^{(1)\,\dagger}(M, \mathbf{\Omega}) \right] \\ + \frac{\alpha_{\rm S}}{\pi} \mathbf{D}^{(1)}(\hat{\mathbf{b}}; M, \mathbf{\Omega}) + \mathcal{O}(\alpha_{\rm S}^2) \,. \qquad (6.21)\end{aligned}$$

The formula in Eq. (5.23), together with the functions C_{ab}, G_{ab}, S_c defined in Chapter 5 and with the hard-virtual amplitude and the soft-real radiative factor defined in this section, gives the NLO singular contribution to the fully-differential cross section of the hadronic scattering in Eq. (6.2). In the following we propose an all-order resummed formula for the hadronic cross section, by combining the BCMN approach to multiparton threshold resummation (employed in Chapter 4) with the BCDG approach to transverse-momentum resummation (presented in Chapter 5).

6.2 All-order resummation

In Chapter 4 we studied a scattering process where four hard partons are present at the Born-level and we discussed the extension of the soft-gluon resummation formulae for the DY and DIS processes presented in Chapter 3, where two hard partons are present at the Born level. According to the BCMN formula, the collinear factors $\Delta_{c,N}$ of Eq. (3.28) and $J_{c,N}$ of Eq. (3.47), defined for the DY and DIS processes, resum all the LL terms that appear in the cross sections of multiparton processes. Already at the NLL, however, the additional colour-space radiative factor $\mathbf{\Delta}_N^{(\rm int)}$ of Eq.(4.33) is required to resum logarithmic terms associated to non-collinear soft-gluon exchanges between the hard partons. The result is the all-order resummation formula in Eq. (4.30).

Along the same lines, the BCDG formula of Eq. (5.23) directly resums all the LL terms of the cross section (6.3) via the Sudakov form factor S_c of Eq. (5.34), but it misses some NLL terms, specifically the ones associated to non-collinear soft-gluon exchanges between the four hard partons, being designed to resum pure collinear effects in DY-like processes. To include the additional logarithmic terms we have introduced the colour-space factor $(\mathbf{H}\,\mathbf{\Delta})_c$ of Eqs. (6.19) and (6.20), that, in analogy to the term $\langle \mathcal{M}_H | \mathbf{\Delta}_N^{(\rm int)} | \mathcal{M}_H \rangle$ in

Eq. (4.30), reproduces colour interferences between real and virtual corrections. To resum the logarithmic terms we have to promote the fixed-order definition of $\mathbf{\Delta}$ in Eq. (6.21) to an expression formally valid to all orders. For simplicity we can first consider the azimuthal average of $\mathbf{\Delta}$ over $\hat{\mathbf{b}}$: being $\langle \mathbf{D}^{(1)} \rangle = 0$ we can write

$$\langle \mathbf{\Delta} \rangle (b; M, \mathbf{\Omega}) = \mathbf{V}^{\dagger}(b; M, \mathbf{\Omega}) \, \mathbf{V}(b; M, \mathbf{\Omega}) \,, \tag{6.22}$$

where

$$\mathbf{V}(b; M, \mathbf{\Omega}) = \exp \left\{ - \int_{b_0^2/b^2}^{M^2} \frac{dk_{\mathrm{T}}^2}{k_{\mathrm{T}}^2} \, \frac{\alpha_{\mathrm{S}}(k_{\mathrm{T}}^2)}{\pi} \, \mathbf{\Gamma}_{\mathrm{T}}^{(1)}(M, \mathbf{\Omega}) \right. \\ \left. + \, \mathcal{O}(\alpha_{\mathrm{S}} \ln(M^2/b^2)) \right\} . \tag{6.23}$$

The expression in Eq. (6.23) resums the NLL terms (i.e. the LL terms due to $\mathbf{\Delta}$), thanks to the choice $\mu^2 = k_{\mathrm{T}}^2$ for the scale of the α_{S} coupling. To reach higher logarithmic accuracy we replace $\mathbf{\Gamma}_{\mathrm{T}}^{(1)}$ with the all-order expression

$$\frac{\alpha_{\mathrm{S}}}{\pi} \, \mathbf{\Gamma}_{\mathrm{T}}^{(1)}(M, \mathbf{\Omega}) \to \mathbf{\Gamma}_{\mathrm{T}}(M, \mathbf{\Omega}; \alpha_{\mathrm{S}}) \,, \tag{6.24}$$

where $\mathbf{\Gamma}_{\mathrm{T}}(\alpha_{\mathrm{S}})$ is a colour-space matrix and it has a perturbative series in α_{S} that plays the same role as $\mathbf{\Gamma}$ in Eq. (4.34). By comparing Eq. (6.22) with Eq. (4.33), it appears evident that the colour operators $\mathbf{\Delta}_N$ and $\langle \mathbf{\Delta} \rangle (b)$ have the same function: they both describe large-angle QCD emission from the scale (M and p_T) typical of the hard process to a soft scale ($1/b$ and p_T/N) associated to a specific constraint (on q_{T} and on z). They depend on the full Born-level kinematics, i.e. on the hard scale and on variables (ω) or r), and additionally on the soft scale. The azimuthal degree-of-freedom of $\mathbf{q}_\perp$ has indeed been integrated out, in defining $\langle \mathbf{\Delta} \rangle$. The $\mathbf{q}_\perp$-distribution (6.3) is however constrained by $\hat{\mathbf{q}}_\perp$ as well and the all-order expression of the colour-space operator $\mathbf{\Delta}(\mathbf{b})$ has a richer structure than $\langle \mathbf{\Delta} \rangle (b)$: it combines the resummed expression in Eq. (6.22) with the additional non-logarithmic $\hat{\mathbf{b}}$-dependent contributions controlled by the colour-space matrix $\mathbf{D}(\hat{\mathbf{b}})$. The all-order expression of the operator $\mathbf{\Delta}$ is given in Eq. (6.28).

We propose the following all-order resummation formula for the singular part of the hadronic cross section (6.3):

$$\frac{d\sigma_{h_1h_2\to Q\bar{Q}+X}(s;\mathbf{q}_\perp,M,y,\mathbf{\Omega})}{d^2\mathbf{q}_\perp\,dM^2\,dy\,d\mathbf{\Omega}} = \frac{M^2}{s}\sum_{c=q,\bar{q},g}[d\sigma^{(0)}_{c\bar{c},F}]\int\frac{d^2\mathbf{b}}{(2\pi)^2}\,e^{i\mathbf{b}\cdot\mathbf{q}_\perp}\,S_c(M,b)$$
$$\times\sum_{a_1,a_2}\int_{x_1}^1\frac{dz_1}{z_1}\int_{x_2}^1\frac{dz_2}{z_2}\,[(\mathbf{H}\,\mathbf{\Delta})\,C_1C_2]_{c\bar{c};a_1a_2}\,f_{a_1/h_1}\!\left(\frac{x_1}{z_1},\frac{b_0^2}{b^2}\right)f_{a_2/h_2}\!\left(\frac{x_2}{z_2},\frac{b_0^2}{b^2}\right), \tag{6.25}$$

where the simbolic factor $[(\mathbf{H}\,\mathbf{\Delta})\,C_1C_2]$, that replaces the factor $\left[H^FC_1C_2\right]$ of Eq. (5.23), is defined as

$$[(\mathbf{H}\,\mathbf{\Delta})\,C_1C_2]_{q\bar{q};a_1a_2} = (\mathbf{H}\,\mathbf{\Delta})_q\;\;C_{ca_1}(z_1;\alpha_S(b_0^2/b^2))\,C_{\bar{c}a_2}(z_2;\alpha_S(b_0^2/b^2))\,, \tag{6.26}$$

in the $q\bar{q}$ channel and

$$[(\mathbf{H}\,\mathbf{\Delta})\,C_1C_2]_{gg;a_1a_2} = (\mathbf{H}\,\mathbf{\Delta})_{g;\mu_1\nu_1,\mu_2\nu_2}$$
$$\times\,C^{\mu_1\nu_1}_{ga_1}(z_1;p_1,p_2,\mathbf{b};\alpha_S(b_0^2/b^2))\,C^{\mu_2\nu_2}_{ga_2}(z_2;p_1,p_2,\mathbf{b};\alpha_S(b_0^2/b^2))\,, \tag{6.27}$$

in the gg channel. Eqs. (6.26) and (6.27) are the analogous of Eqs. (5.30) and (5.54) respectively, with the factor H^F_c replaced by the symbolic factor $(\mathbf{H}\,\mathbf{\Delta})_c$.

The coefficient functions C_{ab}, the gluonic tensors $C^{\mu\nu}_{ga}$ and the Sudakov form factors S_c are defined in Chapter 5. The operator $(\mathbf{H}\,\mathbf{\Delta})_q$ is defined in Eqs. (6.19) and (6.20). The all-order form of $\mathbf{\Delta}$ is

$$\mathbf{\Delta}(\mathbf{b};M,\mathbf{\Omega}) = \mathbf{V}^\dagger(b;M,\mathbf{\Omega})\,\mathbf{D}(\hat{\mathbf{b}};M,\mathbf{\Omega};\alpha_S(b_0^2/b^2))\,\mathbf{V}(b;M,\mathbf{\Omega})\,, \tag{6.28}$$

where

$$\mathbf{V}(b;M,\mathbf{\Omega}) = \bar{P}_q\,\exp\left\{-\int_{b_0^2/b^2}^{M^2}\frac{dq^2}{q^2}\,\mathbf{\Gamma}_\mathrm{T}(M,\mathbf{\Omega};\alpha_S(q^2))\right\}\,. \tag{6.29}$$

The soft-gluon anomalous dimension $\mathbf{\Gamma}_\mathrm{T}(M,\mathbf{\Omega};\alpha_S)$ is a colour-space matrix, and the operator $\bar{P}_q$ denotes inverse q-ordering in the expansion of the exponential matrix. $\mathbf{V}^\dagger$ in Eq. (6.28) is obtained from Eq. (6.29) by replacing $\bar{P}_q$ with P_q, the operator acting in the opposite order. The anomalous-dimension matrix $\mathbf{\Gamma}_\mathrm{T}(\alpha_S)$ and the azimuthal-correlation matrix $\mathbf{D}(\hat{\mathbf{b}};\alpha_S)$ are independent of the details of the hard-scattering process. They are perturbative

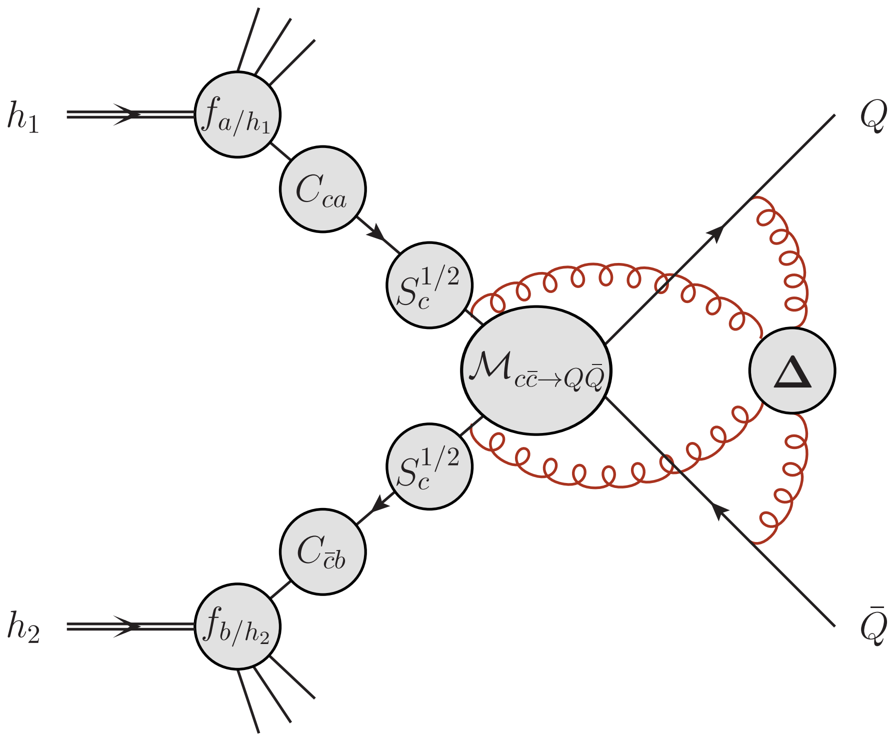

Figure 6.1: Pictorial representation of the resummation formula in Eq. (6.25) for $Q\bar{Q}$ production. The content is the same as in the BCDG formula (Fig. 5.1) with additional large-angle soft-gluon radiation factorised as in the BCMN formula (Fig. 4.1).

series in α_S,

$$\mathbf{\Gamma}_{\mathrm{T}}(M, \mathbf{\Omega}; \alpha_S) = \frac{\alpha_S}{\pi}\, \mathbf{\Gamma}_{\mathrm{T}}^{(1)}(M, \mathbf{\Omega}) + \sum_{n=2}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n \mathbf{\Gamma}_{\mathrm{T}}^{(n)}(M, \mathbf{\Omega})\,, \tag{6.30}$$

$$\mathbf{D}(\hat{\mathbf{b}}; M, \mathbf{\Omega}; \alpha_S) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n \mathbf{D}^{(n)}(\hat{\mathbf{b}}; M, \mathbf{\Omega})\,, \tag{6.31}$$

with first-order terms $\mathbf{\Gamma}_{\mathrm{T}}^{(1)}$ and $\mathbf{D}^{(1)}$ given in Eqs. (6.36) and (6.46).

The all-order expression of $\mathbf{V}$ in Eq. (6.23) follows from Eqs. (6.23) and (6.24), while the definition of $\boldsymbol{\Delta}$ in Eq. (6.28) is an extension of $\langle \boldsymbol{\Delta} \rangle$ in Eq. (6.22) to include the azimuthal-correlation matrix $\mathbf{D}$. The scale of the coupling for $\mathbf{D}$ is set to b_0^2/b^2, in Eq. (6.28): the colour flow described by this operator is due to gluon exchanges at the soft scale and specifically to soft-gluon effects not controlled by the anomalous dimension. The coefficients of the perturbative expansion of $\mathbf{D}$ in powers of $\alpha_S(b_0^2/b^2)$ do not contain logarithmic enhancements in $\ln(M^2b^2)$. However, $\ln(M^2b^2)$ terms are then produced by reexpressing $\alpha_S(b_0^2/b^2)$ in terms of $\alpha_S(M^2)$. Therefore, when transforming back to q_T space, $\mathbf{D}$ contributes as well to logarithmically enhanced terms (see e.g. the last term on the r.h.s. of Eq. (6.14)).

The expression in Eq. (6.28), with $\mathbf{D}$ between $\mathbf{V}^\dagger$ and $\mathbf{V}$, accounts for the impact of azimuthal-correlated emissions on the general structure of the colour-correlation terms. At variance with the cross section for one-particle inclusive production, where only the total energy-fraction z is triggered at the partonic level, we are now able to keep trace not only of the colour correlations but also of the azimuthal correlations and to describe the interplay between the four hard partons in terms of the scale q_T and of the azimuth $\hat{\mathbf{q}}_\perp$. The $[(\mathbf{H}\,\boldsymbol{\Delta})]$ factor has the structure

$$\langle \widetilde{\mathcal{M}}(\alpha_S(M^2))|\, \mathbf{V}^\dagger(b;M)\, \mathbf{D}(\hat{\mathbf{b}}, \alpha_S(b_0^2/b^2))\, \mathbf{V}(b;M)\, |\widetilde{\mathcal{M}}(\alpha_S(M^2))\rangle \qquad (6.32)$$

where the amplitude $|\widetilde{\mathcal{M}}(\alpha_S(M^2))\rangle$ describes the QCD interactions at the hard scale M, the matrix $\mathbf{D}(\hat{\mathbf{b}}, \alpha_S(b_0^2/b^2))$ the interactions at the soft scale $1/b$ and the matrix $\mathbf{V}^\dagger(b;M)$ connects the two different regimes, resumming the ensuing large logarithmic terms.

6.3 Azimuthally-averaged cross section

Starting from the $\mathbf{q}_\perp$-distribution in Eq. (6.25), we can consider the average over the azimuthal degree-of-freedom ($\hat{\mathbf{q}}_\perp \leftrightarrow \phi(\hat{\mathbf{q}}_\perp)$) to define

$$\begin{aligned}\langle \frac{d\sigma_{h_1h_2\to Q\bar{Q}+X}}{d^2\mathbf{q}_\perp\, dM^2\, dy\, d\boldsymbol{\Omega}} \rangle &= \int_0^{2\pi} \frac{d\phi(\mathbf{q}_\perp)}{2\pi}\, \frac{d\sigma_{h_1h_2\to Q\bar{Q}+X}}{d^2\mathbf{q}_\perp\, dM^2\, dy\, d\boldsymbol{\Omega}} \\ &= \frac{1}{\pi}\, \frac{d\sigma_{h_1h_2\to Q\bar{Q}+X}(s; q_T, M, y, \boldsymbol{\Omega})}{dq_T^2\, dM^2\, dy\, d\boldsymbol{\Omega}} \,. \end{aligned} \qquad (6.33)$$

The resummation formula for this cross section is obtained from Eq. (6.25) by replacing the symbolic factor $[(\mathbf{H}\,\boldsymbol{\Delta})\, C_1 C_2]$ with its azimuthal average over ($\hat{\mathbf{b}} \leftrightarrow \phi(\hat{\mathbf{b}})$).

In the $q\bar{q}$ channel, this factor is defined by Eq. (6.26), with the colour-space matrix $\boldsymbol{\Delta}$ being the only term dependent on $\hat{\mathbf{b}}$. The azimuthal average of $\boldsymbol{\Delta}$ is the radiative factor in Eq. (6.22), completely determined by the soft-gluon anomalous-dimension matrix $\boldsymbol{\Gamma}_T$. The azimuthally-averaged cross section in the $q\bar{q}$ channel is thus obtained with $\mathbf{D} \equiv 1$.

For processes initiated by gluon-fusion, the factor in Eq. (6.27) features a more complicated dependence over $\hat{\mathbf{b}}$: the gluonic tensors $C_{ga}^{\mu\nu}$ depends on the azimuth of the transverse momentum $\mathbf{b}$, together with $\boldsymbol{\Delta}$. From the explicit expression of $C_{ga}^{\mu\nu}$, given in Eq. (5.57), we can distinguish between two different contributions: one controlled by the coefficient functions C_{ab}, the other by the coefficient functions G_{ab}. The former reduces to an average of $\boldsymbol{\Delta}$ and yields a result analogous to that of the $q\bar{q}$ channel, with the effective prescription $\mathbf{D} \equiv 1$. The latter contains the azimuthal average of the tensor (and matrix in colour space)

$$D^{\mu_1\nu_1}(p_1, p_2; \mathbf{b})\, \mathbf{D}(\hat{\mathbf{b}}; M, \boldsymbol{\Omega}; \alpha_S(b_0^2/b^2))\, D^{\mu_2\nu_2}(p_1, p_2; \mathbf{b})\,. \tag{6.34}$$

The calculation of the azimuthally averaged transverse-momentum cross section and the resummation of the logarithmically enhanced terms has been independently performed in Refs. [62, 63], by using SCET techniques. The fixed-order cross section that can be extracted from their results is in complete agreement with ours and the same is true for the resummed results, if we remove the matrix $\mathbf{D}(\alpha_S)$ from Eq. (6.28). The resummation formula for the gluon fusion channel demands however a careful handling: the expression in Eq. (6.34) entails azimuthal correlations between initial-state collinear emission and large-angle soft-gluon emission and gives rise to a contribution, specific to processes initiated by gluon fusion, that can not be simply obtained by setting $\mathbf{D} \equiv 1$. In turn, the presence of azimuthal correlations implies the presence of spin correlations [64] and we should not forget that the terms in Eq. (6.34) are matrices in colour space and thus affect the colour flow. A single operator (Lorentz tensor and colour-space matrix) is then responsible for azimuth, spin (helicity) and colour correlations at the same time.

This is of crucial importance in order to understand the structure of the large logarithmic terms: even the azimuthally-averaged cross section is affected by the form of the colour-space matrix $\mathbf{D}(\alpha_S)$, in the gg channel, thanks to a convolution with the Altarelli-Parisi splitting probabilities $\hat{P}_{ga}^{\mu\nu}$ of the initial-state gluons (the $C_{ga}^{\mu\nu}$ tensors in $\mathbf{b}$-space). It follows that our fixed-order calculation of the $\mathbf{q}_\perp$-distribution is essential to deriving the complete transverse-momentum resummation formula, while the simpler q_T-distribution does not include all the relevant information ($\mathbf{D}(\alpha_S)$) and its ensuing contribution. As observed in Section 5.4, the term $G_{ga}(\alpha_S)\, G_{gb}(\alpha_S)$ (i.e. the coefficient of the tensor in Eq. (6.34)) starts at order $\mathcal{O}(\alpha_S^2)$ and it enters the q_T-distribution at the NNLO+NNLL accuracy. The same is true for the colour-space matrix $\mathbf{D}(\alpha_S)$: it does not affect the q_T-distribution up to the NLL accuracy and its first-order coefficient $\mathbf{D}^{(1)}$ controls NNLL terms.

6.4 Extension to NNLL accuracy

Our resummation formula is valid to arbitrary logarithmic accuracy and with the $\mathcal{O}(\alpha_S)$ coefficients in Section 6.6 it can be explicitly worked out up to NLL accuracy (see the end of Section 4.3). In addition we can calculate some contributions to higher accuracy: the diagonal structure inherited from the BCDG formula of Eq. (5.23) is known up to NNLL accuracy (for a review see [102]) and the colour-space structure $(\mathbf{H}\,\mathbf{\Delta})_c$ is organised in a way that makes the extension to the NNLL accuracy simple. An entire class of NNLL terms is produced by interferences between NLL terms in the soft-real radiative factor $\mathbf{\Delta}$ and NLO corrections to the hard-virtual amplitude $|\widetilde{\mathcal{M}}\rangle$. These contributions are controlled by products of the $\mathcal{O}(\alpha_S)$ coefficients provided in Section 6.6.

The complete determination of the NNLL terms require the explicit inclusion of higher-order terms in the anomalous dimension $\mathbf{\Gamma}_{\mathrm{T}}$, in the subtraction operator $\mathbf{I}_{\mathrm{HQ}}$ and in the azimuthal-correlation matrix $\mathbf{D}$. We remind that the phase-space region of non-collinear soft-gluon emission in the low-q_{T} limit corresponds to the near-threshold region ($z_1 = z_2 = 1$), as discussed in the comments to Eq. (6.4). It follows that the second-order coefficient $\mathbf{\Gamma}_{\mathrm{T}}^{(2)}$ can be extracted from the anomalous dimension matrix that controls the threshold logarithms of the $Q\bar{Q}$ invariant mass distribution [126], calculated in [129]. The same argument is employed by the authors of [62, 63] to calculate the soft-functions of their transverse-momentum resummation formula for $t\bar{t}$ production, derived in SCET. The matrix $\mathbf{\Gamma}_{\mathrm{T}}^{(2)}$ also controls the two-loop divergences of $\mathbf{I}_{\mathrm{HQ}}^{(2)}$, together with the first-order coefficients $\mathbf{\Gamma}_{\mathrm{T}}^{(1)}$ and $\mathbf{F}^{(1)}$. The finite terms of $\mathbf{I}_{\mathrm{HQ}}^{(2)}$ are instead determined by a second-order coefficient $\mathbf{F}^{(2)}$, currently unknown.

The colour-space matrices $\mathbf{D}^{(2)}(\hat{\mathbf{b}}; M, \mathbf{\Omega})$ and $\mathbf{F}^{(2)}(M, \mathbf{\Omega})$ must be explicitly calculated and can not be derived from known results. They can be inferred from the singular contribution of the $\mathcal{O}(\alpha_S^2)$ radiative corrections to the $Q\bar{Q}$ transverse-momentum cross section. This fixed-order result is to be compared with our resummation formula, expanded and truncated at NNLO: the logarithmic structure of the two expressions should agree and from the constant terms of the fixed-order calculation we would be able able to determine the $\mathbf{D}^{(2)}$ and $\mathbf{F}^{(2)}$ matrices. We stress that, although these terms are free of explicit dependence on $\ln(M^2b^2)$, they contribute at NNLL accuracy, through the interference with LL terms resummed by the Sudakov form factors S_c (the ones controlled by $A_c^{(1)}$).

To conclude, we note that the calculation of the $\mathcal{O}(\alpha_S^2)$ radiative corrections to the $\mathbf{q}_\perp$-distribution is rather complicated, even in the singular limit $\mathbf{q}_\perp \to 0$, due to the presence of both colour and azimuthal correlations. The fixed-order calculation of the azimuthally-averaged cross section, i.e. of the q_T-distribution, is in contrast definitely simpler, but in this cross section all the information on the azimuthal dependence is lost. However, if we restrict ourselves to the azimuthally-averaged resummed cross section, the second-order coefficient $\mathbf{D}^{(2)}$ first enters at NNNLL accuracy. At NNLL accuracy, the relevant information to include the effect of azimuthal correlations is completely included in the first-order coefficient $\mathbf{D}^{(1)}$, that we have calculated. The colour-space matrix $\mathbf{F}^{(2)}$ is, therefore, the only missing ingredient for the explicit determination of our resummation formula at full NNLO+NNLL accuracy.

6.5 Application to q_T subtraction

The results presented in this Chapter have implications not only for the resummation of the logarithmically enhanced contributions but also for a possible fully differential computation at NLO and NNLO for heavy-quark production and related processes. The problem of performing (fully) differential QCD computation is well known and definitely non-trivial. At NNLO, besides the IR divergent two-loop and one-loop squared contributions, one has to consider the real-virtual corrections with one unresolved parton and the tree-level contributions. To be able to implement these amplitudes into a parton level Monte Carlo program one has to organize the various contributions in a way to explicitly cancel the IR singularities, such that the calculation can be eventually carried out in four dimensions. Various methods have been proposed to achieve this goal [130–134]. The q_T subtraction formalism [133] is a method that is fully developed to work with the production of colourless final states. According to this method the NNLO cross section for the production of a colourless high mass system F can be written as

$$d\sigma^F_{NNLO} = \mathcal{H}^F_{NNLO} \otimes d\sigma^F_{LO} + \left[d\sigma^{F+\text{jet}}_{NLO} - d\sigma^{CT}_{NLO} \right] \ , \tag{6.35}$$

where $d\sigma^{F+\text{jet}}_{NLO}$ is the cross section for the inclusive production of the system F plus one jet at NLO accuracy, and can be evaluated, for example, with the dipole subtraction formalism [66]. When the transverse momentum q_T of the colourless system F is non-vanishing, $d\sigma^{F+\text{jet}}_{NLO}$ is the sole contribution to the NNLO cross section. The IR subtraction counterterm $d\sigma^{CT}_{NLO}$ in

Eq. (6.35) has the purpose of cancelling the singularity developed by $d\sigma^{F+\text{jet}}_{NLO}$ as $q_{\mathrm{T}} \to 0$. To construct the explicit form of $d\sigma^{CT}_{NLO}$, one can use the resummation formula (5.23) discussed in Chapter 5: when this formula is expanded up to $\mathcal{O}(\alpha_{\mathrm{S}}^2)$ it provides a natural candidate for the subtraction counterterm, since it correctly describes the small-q_{T} behavior of the transverse-momentum cross section, and all its coefficients are explicitly know up to this order. The function $\mathcal{H}^F_{NNLO}$, which also compensates for the subtraction of $d\sigma^{CT}_{NLO}$, corresponds to the NNLO truncation of the process-dependent perturbative function $[H^F C_1 C_2]_{c\bar{c}a_1a_2}$ that controls q_{T} resummation for the related transverse-momentum cross section at NNLL+NNLO accuracy. As discussed in Chapter 5, the general structure of the hard function is known up to NNLO [92,102–106], and thus Eq. (6.35) can be numerically implemented into a parton level event generator, once the corresponding amplitudes are available. The q_{T} subtraction method has been successfully applied to a variety of relevant hadronic processes: single Higgs [133] and vector boson production [135], associated WH production [136], photon pairs [137], $Z\gamma$ [138], $W\gamma$ [139] and ZZ production [140]. In its current formulation, however, the method is developed only for the case in which F is a system of colourless particles.

The results presented in this Chapter offer the possibility to extend the q_{T} subtraction method to the production of a $Q\bar{Q}$ pair. Since the small-q_{T} behavior of the $Q\bar{Q}$ cross section up to $\mathcal{O}(\alpha_{\mathrm{S}})$ is known in explicit form, by including the $\delta(q_{\mathrm{T}}^2)$ term (see Sections 6.1 and 6.6), the subtraction counterterm and the relevant hard coefficient appearing in Eq. (6.35) can be explicitly constructed. We have implemented [141] all these ingredients in a numerical program based on Eq. (6.35) and we have checked that we can reproduce the numerical results produced with the general purpose NLO generator MCFM. The extension to NNLO is of course non-trivial and requires the computation of the second-order coefficient $\mathbf{F}^{(2)}$.

6.6 First-order coefficients of the colour operators

In the following we give the explicit expressions of the four colour-space operators that control our NLO results in $\mathbf{q}_\perp$-space and in $\mathbf{b}$-space and the resummation formula up to NLL accuracy. The dependence over the Born-level kinematics, $(M, \mathbf{\Omega}) \leftrightarrow (p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu)$, is understood and only the eventual dependence over $\mathbf{q}_\perp$, $\mathbf{b}$ is specified. The momentum p_j^μ ($j = 3, 4$) of the heavy quark is specified by the heavy-quark mass m_j ($p_j^2 = m_j^2$), rapidity y_j and transverse-momentum vector $\mathbf{p}_{j\perp}$.

The operator $\mathbf{\Gamma}_\mathrm{T}^{(1)}$ is

$$\mathbf{\Gamma}_\mathrm{T}^{(1)} = -\frac{1}{4}\Bigg\{ \sum_{j=3,4} \mathbf{T}_j^2\,(1 - i\pi) + \sum_{\substack{i=1,2\\ j=3,4}} \mathbf{T}_i \cdot \mathbf{T}_j\, \ln \frac{(2\,p_i \cdot p_j)^2}{M^2 m_j^2} \\ + \mathbf{T}_3 \cdot \mathbf{T}_4 \left[\frac{1}{v_{34}} \ln \frac{1 + v_{34}}{1 - v_{34}} - 2\,i\pi \left(\frac{1}{v_{34}} + 1 \right) \right] \Bigg\}, \tag{6.36}$$

where v_{34} is the relative velocity of the heavy-quark pair:

$$v_{jk} = \sqrt{1 - \frac{m_j^2 m_k^2}{(p_j \cdot p_k)^2}}\,. \tag{6.37}$$

The operator $\mathbf{F}^{(1)}$ is

$$\mathbf{F}^{(1)} = \sum_{j=3,4} \mathbf{T}_j^2\, \ln \frac{m_j^2 + \mathbf{p}_{j\perp}^2}{m_j^2} - \sum_{\substack{i=1,2\\ j=3,4}} \mathbf{T}_i \cdot \mathbf{T}_j\, \mathrm{Li}_2\left(-\frac{\mathbf{p}_{j\perp}^2}{m_j^2} \right) + \mathbf{T}_3 \cdot \mathbf{T}_4\, \frac{F_{34}}{v_{34}}\,, \tag{6.38}$$

where

$$F_{jk} = \frac{1}{2} \ln \frac{1 + v_{jk}}{1 - v_{jk}}\, \ln \frac{(m_j^2 + \mathbf{p}_{j\perp}^2)(m_k^2 + \mathbf{p}_{k\perp}^2)}{m_j^2\, m_k^2} - \frac{1}{4} \ln^2 \frac{1 + v_{jk}}{1 - v_{jk}} \\ - 2\,\mathrm{Li}_2\left(\frac{2\,v_{jk}}{1 + v_{jk}} \right) + \sum_{i=1,2} \left[\mathrm{Li}_2\left(1 - \sqrt{\frac{1 - v_{jk}}{1 + v_{jk}}}\; r_{jk,i} \right) \right. \\ \left. +\ \mathrm{Li}_2\left(1 - \sqrt{\frac{1 - v_{jk}}{1 + v_{jk}}}\; \frac{1}{r_{jk,i}} \right) + \frac{1}{2} \ln^2 r_{jk,i} \right], \tag{6.39}$$

and

$$r_{jk,i} = \frac{m_k}{m_j} \frac{p_i \cdot p_j}{p_i \cdot p_k}\,. \tag{6.40}$$

In order to define $\mathbf{R}^{(1)}$ and $\mathbf{D}^{(1)}$, we express the azimuthal degrees-of-freedom in $\mathbf{q}_\perp$-space and in $\mathbf{b}$-space in terms of the unitary vectors

$$\hat{\mathbf{q}}_\perp = \frac{\mathbf{q}_\perp}{\sqrt{\mathbf{q}_\perp^2}}\,, \qquad \hat{\mathbf{b}} = \frac{\mathbf{b}}{\sqrt{\mathbf{b}^2}}\,, \tag{6.41}$$

and the projections

$$\hat{\mathbf{q}}_j = \frac{\hat{\mathbf{q}}_\perp \cdot \mathbf{p}_{j\perp}}{\sqrt{m_j^2 + \mathbf{p}_{j\perp}^2}}\,, \qquad \hat{\mathbf{b}}_j = \frac{\hat{\mathbf{b}} \cdot \mathbf{p}_{j\perp}}{m_j}\,. \tag{6.42}$$

The operator $\mathbf{R}^{(1)}$ is

$$\begin{aligned}\mathbf{R}^{(1)}(\hat{\mathbf{q}}_\perp) = -\frac{1}{2}\Bigg\{ & \mathbf{T}_3 \cdot \mathbf{T}_4\, R_{34}(\hat{\mathbf{q}}_\perp) \\ & + \sum_{\substack{i=1,2\\ j=3,4}} \mathbf{T}_i \cdot \mathbf{T}_j \left[\ln \frac{(p_i \cdot p_j)^2}{(p_1 \cdot p_j)(p_2 \cdot p_j)} + \frac{2\,\hat{\mathbf{q}}_j}{\sqrt{1-\hat{\mathbf{q}}_j^2}} \left(\arctan \frac{\hat{\mathbf{q}}_j}{\sqrt{1-\hat{\mathbf{q}}_j^2}} - \frac{\pi}{2} \right) \right] \\ & + \sum_{j=3,4} \mathbf{T}_j^2\, \frac{m_j^2}{m_j^2 + \mathbf{p}_{j\perp}^2} \frac{1}{1-\hat{\mathbf{q}}_j^2} \left[1 + \frac{\hat{\mathbf{q}}_j}{\sqrt{1-\hat{\mathbf{q}}_j^2}} \left(\arctan \frac{\hat{\mathbf{q}}_j}{\sqrt{1-\hat{\mathbf{q}}_j^2}} - \frac{\pi}{2} \right) \right] \Bigg\}\,,\end{aligned} \tag{6.43}$$

where

$$\begin{aligned} R_{jk}(\hat{\mathbf{q}}_\perp) = & \frac{p_j \cdot p_k}{\sqrt{(m_j^2 + \mathbf{p}_{j\perp}^2)(m_k^2 + \mathbf{p}_{k\perp}^2)}} \frac{1}{(\hat{\mathbf{q}}_j^2 + \hat{\mathbf{q}}_k^2) - 2\,\hat{\mathbf{q}}_j t_{jk} \hat{\mathbf{q}}_k + (t_{jk}^2 - 1)} \\ & \times \Bigg[\frac{(p_1 \cdot p_j)(p_2 \cdot p_k)}{(p_1 \cdot p_2)\sqrt{(m_j^2 + \mathbf{p}_{j\perp}^2)(m_k^2 + \mathbf{p}_{k\perp}^2)}} \ln \frac{(p_1 \cdot p_j)(p_2 \cdot p_k)}{(p_2 \cdot p_j)(p_1 \cdot p_k)} \\ & + \frac{2\,(\hat{\mathbf{q}}_j t_{jk} - \hat{\mathbf{q}}_k)}{\sqrt{1-\hat{\mathbf{q}}_j^2}} \left(\arctan \frac{\hat{\mathbf{q}}_j}{\sqrt{1-\hat{\mathbf{q}}_j^2}} - \frac{\pi}{2} \right) + (j \leftrightarrow k) \Bigg]\,, \end{aligned} \tag{6.44}$$

and

$$t_{jk} = \frac{p_j \cdot p_k + \mathbf{p}_{j\perp} \cdot \mathbf{p}_{k\perp}}{\sqrt{(m_j^2 + \mathbf{p}_{j\perp}^2)(m_k^2 + \mathbf{p}_{k\perp}^2)}} = \frac{(p_1 \cdot p_j)(p_2 \cdot p_k) + (p_2 \cdot p_j)(p_1 \cdot p_k)}{(p_1 \cdot p_2)\sqrt{(m_j^2 + \mathbf{p}_{j\perp}^2)(m_k^2 + \mathbf{p}_{k\perp}^2)}}\,. \tag{6.45}$$

The azimuthal average of $\mathbf{R}^{(1)}$ is $\langle \mathbf{R}^{(1)} \rangle = \left[\mathbf{\Gamma}_{\mathrm{T}}^{(1)} + \mathbf{\Gamma}_{\mathrm{T}}^{(1)\,\dagger} \right]$, from Eq. (6.16). The operator $\mathbf{D}^{(1)}$ can be written as the difference of an operator $\mathbf{E}^{(1)}$ and its azimuthal average $\langle \mathbf{E}^{(1)} \rangle$ over $\hat{\mathbf{b}}$:

$$\mathbf{D}^{(1)}(\hat{\mathbf{b}}) = \frac{1}{2}\left[\mathbf{E}^{(1)}(\hat{\mathbf{b}}) - \langle \mathbf{E}^{(1)} \rangle \right] = \frac{1}{2}\left[\mathbf{E}^{(1)}(\hat{\mathbf{b}}) - \mathbf{F}^{(1)} \right]\,. \tag{6.46}$$

The azimuthal average of $\mathbf{E}^{(1)}$ is $\langle \mathbf{E}^{(1)} \rangle = \mathbf{F}^{(1)}$. The operator $\mathbf{E}^{(1)}$ is

$$\begin{aligned}\mathbf{E}^{(1)}(\hat{\mathbf{b}}) = \mathbf{T}_3 \cdot \mathbf{T}_4 \, \frac{E_{34}(\hat{\mathbf{b}})}{v_{34}} + \sum_{j=3,4} \mathbf{T}_j^2 \, \frac{\hat{\mathbf{b}}_j}{\sqrt{1+\hat{\mathbf{b}}_j^2}} \, \Big[2\,\mathrm{arcsinh}(\hat{\mathbf{b}}_j) - i\pi \Big] \\ + \sum_{\substack{i=1,2 \\ j=3,4}} \mathbf{T}_i \cdot \mathbf{T}_j \; 2\,\mathrm{arcsinh}(\hat{\mathbf{b}}_j) \Big[\mathrm{arcsinh}(\hat{\mathbf{b}}_j) - i\pi \Big] \, , \end{aligned} \tag{6.47}$$

where

$$E_{jk}(\hat{\mathbf{b}}) = \mathrm{sign}(\hat{\mathbf{b}}_j - \hat{\mathbf{b}}_k \gamma_{jk}) \Big[L_z(\hat{\mathbf{z}}_{jk}, \hat{\mathbf{c}}_{jk}) - L_z(\hat{\mathbf{b}}_k, \hat{\mathbf{c}}_{jk}) \Big] + (j \leftrightarrow k) \, , \tag{6.48}$$

and

$$\begin{aligned} L_z(z,c) &= L_\xi\big(\xi(z,c), c\big) \, , \quad \xi(z,c) = \Big(z + \sqrt{1+z^2} \Big) \Big(z + \sqrt{c+z^2} \Big) \, , \\ L_\xi(\xi, c) &= \frac{1}{2} \ln^2 \frac{\xi(1+\xi)}{c+\xi} - \ln^2 \frac{\xi}{c+\xi} - i\pi \ln \frac{(c+\xi)(1+\xi)}{\xi} \\ &\quad + 2 \left[\mathrm{Li}_2(-\xi) - \mathrm{Li}_2 \frac{c+\xi}{c-1} - \ln(c+\xi) \ln \frac{1}{1-c} \right] , \end{aligned} \tag{6.49}$$

$$\begin{aligned} \hat{\mathbf{z}}_{jk} &= \mathrm{sign} \Big[\hat{\mathbf{b}}_j \, (\gamma_{jk} m_j - m_k) + \hat{\mathbf{b}}_k \, (\gamma_{jk} m_k - m_j) \Big] \sqrt{-\hat{\mathbf{c}}_{jk}} \, , \\ \hat{\mathbf{c}}_{jk} &= \frac{\hat{\mathbf{b}}_j^2 - 2\,\hat{\mathbf{b}}_j \gamma_{jk} \hat{\mathbf{b}}_k + \hat{\mathbf{b}}_k^2}{\gamma_{jk}^2 - 1} \, , \qquad \gamma_{jk} = \frac{1}{\sqrt{1 - v_{jk}^2}} = \frac{p_j \cdot p_k}{\sqrt{m_j^2 m_k^2}} \, . \end{aligned} \tag{6.50}$$

Chapter 7

Conclusions

The Large Hadron Collider is exploring the interactions of the fundamental particles at unprecedented energy scales. In order to perform precise tests of the Standard Model and to disentangle possible (small) new physics effects, accurate theoretical predictions are necessary. Our capability to obtain such predictions for hard-scattering processes at hadron colliders is based on the factorisation theorem, which allows us to write the hadronic cross section for a hard-scattering process as a convolution of parton distribution functions, that are extracted from global fits to experimental data, with the hard-scattering partonic cross section, that can be computed in perturbation theory. The possibility to organise the hard-scattering cross section as an expansion in the QCD coupling α_S allowed to perform accurate calculations up to NLO and, for some specific processes, to the NNLO.

Fixed-order calculations, however, are reliable only if the measured energy scales are all of the same order. Near the exclusive boundaries of the phase space, the perturbative expansion is plagued by large logarithmic terms, of infrared nature, which must be summed to all orders. Such resummation is effectively performed by Monte Carlo event generators, though with very limited logarithmic accuracy. In particular, processes initiated by multiparton scatterings at tree level require a dedicated treatment. In this case the complicated colour flow features a large-angle contribution that is irreducible, i.e. that can not be reduced to a sum of colour-uncorrelated collinear emissions from independent QCD emitters, and leads to a non-trivial pattern of soft enhancements. This thesis dealt with a class of processes initiated by the hard scattering of *four* partons at Born level: we have considered the one-particle inclusive cross section at high-transverse energies [83] and heavy-quark pair production at small transverse momentum in hadronic collisions [119]. These processes are particularly relevant for high-energy phenomenology. As regards the production of one hadron at high transverse momentum, previous studies [142–145] have shown that fixed-order results significally underestimate the data (independently collected at DESY, Tevatron, BNL-RHIC, HERA, LEP2), so that the resummation of the large logarithmic terms is crucial [82]. A better estimation of the short-distance cross section translates to stronger constraints on universal long-distance effects, specifically power-suppressed contributions and fragmentation functions for the observed hadron. The production of heavy quark pair is even more relevant at the LHC, given the huge amount of $t\bar{t}$ pairs that have been collected in the $\sqrt{s} = 7$ and 8 TeV runs at the LHC, and the even higher number of $t\bar{t}$ pairs that is expected at Run2. First measurements of the transverse momentum distribution of $t\bar{t}$ pairs from ATLAS and CMS are starting to appear [117, 118].

Our approach to study these processes was based on the use of the factorisation properties of QCD radiation in soft and collinear limits. By devising suitable approximations of the real emission matrix elements, we were able to analytically compute the structure of the logarithmically enhanced terms, down to the constant term that survives in the Sudakov region at the first relative order in α_S, and to explicitly determine the anomalous dimensions that control the pattern of soft-gluon radiation at large angles. These calculations allowed us to write down the corresponding resummation formulae of the large logarithmic contributions, and to determine the coefficients controlling the resummation at full NLL accuracy. The dynamical and kinematical information on the perturbative QCD interactions is explicitely worked out (integrated) up to the level of the measured hadronic variables, whereas the traces of $SU(3)$ generators are intentionally retained, in a way that preserves the full information on the colour flow. The cross sections are in fact factorised in colour space, in terms of projections of colour-space matrices over colour-space amplitudes. The amplitudes capture the hard-scale physics, corresponding to non-soft contributions to multi-loop virtual corrections, while the matrices both account for real and virtual emissions at the soft scale and describe the evolution from the hard to the soft scale, thus resumming the logarithmic terms, enhanced by the strong ordering between the two scales. The separation of hard and soft QCD effects and the adoption of the colour-space formalism has practical implications when it comes to the actual computation of resummed cross sections: the explicit form of the various pieces that compose our resummation formulae at NLL accuracy in addition determines some NNLL contributions, thanks to quantum interference effects. It follows that our approach is suitable for a relatively straightforward extension of resummation to higher logarithmic accuracy, in that the only terms to be computed at NNLL are the ones originated from pure NNLO effects.

In the case of the one-particle cross section, we have checked that the fixed-order expansion of the cross section presented in Chapter 4 reproduces the NLO results available in the literature, and that the resummation formula is consistent with less general resummation formulae, valid for the cross section integrated over rapidity or for the hadroproduction of a photon. We have explicitely discussed how the same results can be improved to include NNLL terms. The essential ingredient is the second-order soft-gluon anomalous dimension, for which we have provided a first rough estimation. In addition the hard-virtual amplitude mut be calculated at NNLO, via the inclusion of two-loop corrections and of the second-order coefficient of the subtraction operator.

In the case of $t\bar{t}$ production the results presented in Chapter 6 have implications both in *resummed* and in *fixed order* calculations. The calculation of the $t\bar{t}$ cross section at small transverse momentum provides in fact a consistent framework for the resummation of the $t\bar{t}$ spectrum by confirming results previously obtained in SCET and extending them to include the azimuthal correlations in the small q_{T} limit. At the same time, the results of Chapter 6 pave the way to an extension of the q_{T} subtraction formalism to this class of processes [141]. The q_{T} subtraction formalism [133] is a powerful method to perform fully-exclusive NNLO computations for the class of processes in which a colour singlet final state is produced in hadronic collisions. However, its present formulation cannot be straightforwardly extended to the processes considered in this thesis. As a first concrete application of our results we have applied the q_{T} subtraction to $t\bar{t}$ production at NLO.

The extension of the results of this thesis to higher perturbative orders is certainly non-trivial, but the methods we have used are sufficiently general to make us confident that they can have a number of additional applications.

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